Master of Science in Mathematics
(M.Sc. Mathematics)

Differential Equations
(DMSMCO102T24)

Self-Learning Material
(SEM 1)

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Course Code: DMSMCO102T24
Differential Equations

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COURSE INTRODUCTION

Differential equations, focusing on methods of solution, analysis, and application. Students will explore various types of differential equations, including ordinary differential equations (ODEs) and partial differential equations (PDEs), and learn how to apply them to model real-world phenomena in engineering, physics, biology, and other sciences.

The course is of four credits and is divided into 12 units. Each Unit is divided into sub topics. The course provides an in-depth study of differential equations, which are fundamental to modeling and understanding phenomena in engineering, physics, biology, economics, and other sciences. The course covers the theory, solution techniques, and applications of ordinary differential equations (ODEs) and partial differential equations (PDEs).

Each unit starts with a statement of objectives that outlines the goals we hope you will accomplish.

Course Outcomes:

At the completion of the course, a student will be able to:

1. Recall the derivation of the differential equation, Linear partial differential.
2. Explain Methods of Solving Differential equations of first order.
3. Apply Lagrange’s linear equation, Lagrange’s solution of the linear equation.
4. Analyze Geometrical interpretation of Lagrange’s linear equation.
5. Evaluate the linear equations with n independent variables, special types of equations.
6. Create the Nonlinear PDE of first order, solve using Charpit’s method.

Acknowledgements:

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UNIT - 1

Differential Equations (ODEs)

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure:

1.1 Introduction to Ordinary Differential Equations (ODEs)

1.2 Basic Concepts and Definitions

1.3 Classification of ODEs

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1.1 Introduction to Ordinary Differential Equations (ODEs)

Ordinary Differential Equations (ODEs) are equations that involve functions and their derivatives. They play a crucial role in modeling the behavior of physical systems, biological processes, engineering problems, and many other scientific applications. This introduction will cover the basic concepts, types, and methods of solving ODEs.

1.2 Basic Concepts and Definitions

An equation which involves differential co-efficient is called a differential equation.
For Example,

1. $\frac{dy}{dx} = \frac{1+x^2}{1-y^2}$ 2. $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 8y = 0$ 3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{d^2y}{dx^2}$

4. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu,$ 5. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial y}$

Order and degree of differential equations

The order of a differential is the order of the highest differential co- efficient present in the equation and degree of a differential equation is the degree of the highest derivative after removing the radical sign and fraction.

1. $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin wt.$ 2. $\cos x\frac{d^2y}{dx^2} + \sin x\left(\frac{dy}{dx}\right)^2 + 8y = \tan x$

3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$

The order of the above equations is 2.
The degree of the equation (1) and (2) is 1.
The degree of the equation (3) is 2.

1.3 Classification of ODEs

Ordinary Differential Equations (ODEs) can be classified based on various properties, such as order, linearity, and whether they are homogeneous or non homogeneous. Here's a classification based on these criteria:

First-Order ODEs: The highest derivative in the equation is the first derivative.

Second-Order ODEs: The highest derivative in the equation is the second derivative.

Higher-order ODEs: Similarly, you can have third-order, fourth-order, and so on.

1.4 Homogeneous and Exact Forms of Second Order ODEs

A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = X \qquad \dots(1)$$

is called a homogenous equation.

Then (1) can be written in the form

$$\{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n\} y = X$$

i.e., $$f(D) y = X \qquad \dots(2)$$

where $$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots a_n.$$

Here $f(D)$ is a polynomial of degree n in D

If $x = 0$, the equation

$$f(D) y = 0$$

If x ≠ 0 then the equation 2 is called a non homogenous equation

1.5 Summary

Differential equations are mathematical formulas that use derivatives to explain the relationship between a function and its derivatives. Ordinary differential equations (ODEs) are used to solve functions with a single variable. ODEs are created by expressing the derivatives of an unknown function in terms of the independent variable and the function itself. ODEs of first order and first degree involve the first derivative of an unknown function. Variable separable is a method for solving ODEs in which the variables may be separated on either side of the problem. It entails decoupling the variables, integrating both sides, and arriving at a general solution.

1.6 Keywords

- Ordinary differential equations (ODEs).
- Variable separable

1.7 Self-Assessment Questions

1. Solve:

$$\frac{dy}{dx} = f(x) \Rightarrow \quad y = \int f(x)dx + c$$

2. Solve:

$$\frac{dy}{dx} = f(ax + by + c), \text{ then put } ax + by + c = v$$

3. Solve:

$$\text{If } \frac{dy}{dx} = f(x)g(y) \Rightarrow g(y)^{-1}dy = f(x)dx \text{ then } \int (g(y))^{-1}dy = \int f(x)dx$$

1.8 Case Study

A is a biologist who is researching the population dynamics of a certain species in an ecosystem. The species' population is influenced by a variety of factors, including birth rate, mortality rate, and accessible resources. Your objective is to use differential equations to simulate population increase and analyse population behaviour over time.

Question

1. Given the birth rate of the species as 0.05 individuals per day and the mortality rate as 0.03 individuals per day, calculate the net population growth rate per day.
2. Suppose the accessible resources for the species decrease over time, causing a decline in the birth rate from 0.05 to 0.03 individuals per day. If the mortality rate remains constant at 0.03 individuals per day, calculate the new net population growth rate and the equilibrium population size assuming the birth and mortality rates remain constant.

1.9 References

1. Grewal . B.S., "Elementary Engineering Mathematics", Khanna publications 34th Ed., 1998.
2. Gupta, S. P and Kapoor V.K, Fundamental of Mathematical Statistics, Sultan Chand and Sons, New Delhi.

UNIT - 2

Second Order ODEs

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure:

2.1 Solution Techniques for Second Order ODEs

We consider the homogenous equation

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + qy = 0 \qquad ...(1)$$

There are 3 cases:

Case 1: Roots are real and distinct

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case 2: Roots are equal

$$y = (C_1 + C_2 x) e^{mx}$$

Case 3: Roots are Complex

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

1. Solve:

$$\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

Solution. Given equation is $(D^2 - 5D + 6)\, y = 0$

A.E. is $m^2 - 5m + 6 = 0$

i.e., $(m-2)(m-3) = 0$

i.e., $m = 2, 3$

$\therefore$ $m_1 = 2, m_2 = 3$

$\therefore$ **The roots are real and distinct.**

$$y = C_1 e^{2x} + C_2 e^{3x}.$$

2. Solve:

$$\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

Solution. Given equation is $(D^3 - D^2 - 4D + 4)\, y = 0$

A.E. is $m^3 - m^2 - 4m + 4 = 0$

$m^2 (m - 1) - 4 (m - 1) = 0$

$(m - 1)(m^2 - 4) = 0$

$m = 1, m = \pm 2$

$m_1 = 1, m_2 = 2, m_3 = -2$

$\therefore$ The general solution of the given equation is

$$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-2x}.$$

3. Solve:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0.$$

Solution. The D.E. can be written as

$$(D^2 - D - 6)\,y = 0$$

A.E. is $m^2 - m - 6 = 0$

$\therefore$ $(m - 3)(m + 2) = 0$

$\therefore$ $m = 3, -2$

$\therefore$ The general solution is

$$y = C_1 e^{3x} + C_2 e^{-2x}.$$

4. Solve:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0.$$

Solution. The D.E. can be written as

$$(D^2 + 8D + 16)\,y = 0$$

A.E. is $m^2 + 8m + 16 = 0$

$\therefore$ $(m + 4)^2 = 0$

$(m + 4)(m + 4) = 0$

$m = -4, -4$

$\therefore$ The general solution is

$$y = (C_1 + C_2 x)\,e^{-4x}.$$

Exercise

Solve the following:

1. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = 0.$ $\left[\textbf{Ans. } y = C_1 e^{3x} + C_2 e^{-x}\right]$

2. $6y'' - y' - y = 0.$ $\left[\textbf{Ans. } y = C_1 e^{\frac{1}{2}x} + C_2 e^{-\frac{1}{3}x}\right]$

3. $(2D^2 - D - 6)\,y = 0.$ $\left[\textbf{Ans. } y = C_1 e^{2x} + C_2 e^{-\frac{3}{2}x}\right]$

4. $(D^2 + 4D + 4)\,y = 0.$ $\left[\textbf{Ans. } y = (C_1 + C_2 x)\,e^{-2x}\right]$

5. $9y'' - 6y' + y = 0.$ $\left[\textbf{Ans. } y = (C_1 + C_2 x)\,e^{\frac{1}{3}x}\right]$

6. $y'' + 9y = 0.$ $\left[\textbf{Ans. } y = C_1 \cos 3x + C_2 \sin 3x\right]$

7. $(D^2 - 2D + 2)\,y = 0.$ $\left[\textbf{Ans. } y = e^{x}(C_1 \cos x + C_2 \sin x)\right]$

2.2 Inverse differential operator and particular integral

Consider a differential equation

$$f(D)\, y = x \qquad ...(1)$$

Define $\frac{1}{f(D)}$ such that

$$f(D)\left\{\frac{1}{f(D)}\right\} x = x \qquad ...(2)$$

Here $f(D)$ is called the inverse differential operator. Hence from Eqn. (1), we obtain

$$y = \frac{1}{f(D)} x \qquad ...(3)$$

Since this satisfies the Eqn. (1) hence the particular integral of Eqn. (1) is given by Eqn. (3)

Thus, particular Integral (P.I.) = $\frac{1}{f(D)} x$

The inverse differential operator $\frac{1}{f(D)}$ is linear.

i.e., $\frac{1}{f(D)}\{ax_1 + bx_2\} = a\frac{1}{f(D)} x_1 + b\frac{1}{f(D)} x_2$

where a, b are constants and x_1 and x_2 are some functions of x.

Type 1: P.I. of the form $\frac{e^{ax}}{f(D)}$

We have the equation $f(D)\, y = e^{ax}$

Let $f(D) = D^2 + a_1 D + a_2$

We have $D(e^{ax}) = a\, e^{ax}$, $D^2(e^{ax}) = a^2\, e^{ax}$ and so on.

$\therefore$ $f(D)\, e^{ax} = (D^2 + a_1 D + a_2)\, e^{ax}$

$= a^2 e^{ax} + a_1 . ae^{ax} + a_2 e^{ax}$

$= (a^2 + a_1 . a + a_2)\, e^{ax} = f(a)\, e^{ax}$

Thus $f(b)\, e^{ax} = f(a)\, e^{ax}$

Operating with $\frac{1}{f(D)}$ on both sides

We get, $$e^{ax} = f(a) \cdot \frac{1}{f(D)} \cdot e^{ax}$$

or $$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(D)}$$

In particular if $f(D) = D - a$, then using the general formula.

We get, $$\frac{1}{D-a} e^{ax} = \frac{e^{ax}}{(D-a)\phi(D)} = \frac{1}{D-a} \cdot \frac{e^{ax}}{\phi(a)}$$

i.e., $$\frac{e^{ax}}{f(D)} = \frac{1}{\phi(a)} e^{ax} \int 1 . d\,x = \frac{1}{\phi(a)} . x\, e^{ax}$$

$\therefore$ $$f'(a) = 0 + \phi(a)$$

or $$f'(a) = \phi(a)$$

$$\frac{e^{ax}}{f(D)} = x \cdot \frac{e^{ax}}{f'(D)}$$

where $$f(a) = 0$$

and $$f'(a) \neq 0$$

This result can be extended further also if

$$f'(a) = 0, \quad \frac{e^{ax}}{f(D)} = x^2 . \frac{e^{ax}}{f''(a)} \text{ and so on.}$$

Type 2: P.I. of the form $\dfrac{\sin ax}{f(D)}, \dfrac{\cos ax}{f(D)}$

We have $$D (\sin ax) = a \cos ax$$

$$\begin{aligned} D^2 (\sin ax) &= -a^2 \sin ax \\ D^3 (\sin ax) &= -a^3 \cos ax \\ D^4 (\sin ax) &= a^4 \sin ax \\ &= (-a^2)^2 \sin ax \text{ and so on.} \end{aligned}$$

Therefore, if $f(D^2)$ is a rational integral function of D^2 then $f(D^2) \sin ax = f(-a^2) \sin ax$.

Hence $$\frac{1}{f(D^2)} \left\{ f(D^2) \sin ax \right\} = \frac{1}{f(D^2)} f(-a^2) \sin ax$$

i.e., $$\sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax$$

i.e., $$\frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)}$$

Provided $$f(-a^2) \neq 0 \qquad \text{...(1)}$$

Similarly, we can prove that

$$\frac{1}{f(D^2)}\cos ax = \frac{\cos ax}{f(-a^2)}$$

if $f(-a^2) \neq 0$

In general, $$\frac{1}{f(D^2)}\cos ax = \frac{\cos ax}{f(-a^2)}$$

if $f(-a^2) \neq 0$ ...(2)

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$$

and $$\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b)$$

These formula can be easily remembered as follows.

$$\frac{1}{D^2+a^2}\sin ax = \frac{x}{2}\int \sin ax\,dx = \frac{-x}{2a}\cos ax$$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2}\int \cos ax\,ax = \frac{x}{2a}\sin ax.$$

Type 3: P.I. of the form $\frac{\phi(x)}{f(D)}$ where $\phi(x)$ is a polynomial in x, we seeking the polynomial Eqn. as the particular solution of

$$f(D)y = \phi(x)$$

where $$\phi(x) = a_0 x^n + a_1 x^{n-1} + \ldots a_{n-1} x + a_n$$

Type 4: P.I. of the form $\frac{e^{ax} V}{f(D)}$ where V is a function of x.

We shall prove that $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$.

Consider $D(e^{ax} V) = e^{ax} DV + Va\, e^{ax}$

$= e^{ax}(D + a) V$

and $D^2(e^{ax} V) = e^{ax} D^2 V + a\, e^{ax} DV + a^2 e^{ax} V + a\, e^{ax} DV$

$= e^{ax}(D^2 V + 2a\, DV + a^2 V)$

$= e^{ax}(D + a)^2 V$

Similarly, $D^3(e^{ax} V) = e^{ax}(D + a)^3 V$ and so on.

$\therefore$ $f(D)\, e^{ax} V = e^{ax} f(D + a) V$ ...(1)

Let $f(D + a) V = U$, so that $V = \frac{1}{f(D+a)} U$

Hence (1) reduces to

$$f(D)\, e^{ax}\, \frac{1}{f(D+a)} U = e^{ax}\, U$$

Operating both sides by $\frac{1}{f(D)}$ we get,

$$e^{ax} \frac{1}{f(D+a)} U = \frac{1}{f(D)} e^{ax}\, U$$

i.e.,

$$\frac{1}{f(D)} e^{ax}\, U = e^{ax} \frac{1}{f(D+a)} U$$

Replacing U by V, we get the required result.

Type 5: P.I. of the form $\frac{x\,V}{f(D)}, \frac{x^n\,V}{f(D)}$ where V is a function of x.

By Leibniz's theorem, we have

$$D^n (x\,V) = x\, D^n\, V + n\cdot 1\;\; D^{n-1}\cdot V$$

$$= x\, D^n\, V + \left\{\frac{d}{dD} D^n\right\} V$$

$$\therefore \qquad f(D)\; x\, V = x\, f(D)\;\, V + f'\,(D)\;\, V \qquad ...(1)$$

Eqn. (1) reduces to

$$\frac{x\,V}{f(D)} = \left[x - \frac{f'(D)}{f(D)}\right] \frac{V}{f(D)} \qquad ...(2)$$

1. Solve:

$$\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{5x}.$$

Solution. We have

$$(D^2 - 5D + 6)\, y = e^{5x}$$

A.E. is $m^2 - 5m + 6 = 0$

i.e., $(m - 2)(m - 3) = 0$

$\Rightarrow$ $m = 2, 3$

Hence the complementary function is

$\therefore$ $\text{C.F.} = C_1 e^{2x} + C_2 e^{3x}$

Particular Integral (P.I.) is

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^{5x} \qquad (D \to 5)$$

$$= \frac{1}{5^2 - 5 \times 5 + 6} e^{5x} = \frac{e^{5x}}{6}.$$

$\therefore$ The general solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{3x} + \frac{e^{5x}}{6}.$$

2.Solve:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10e^{3x}.$$

Solution. We have

$$(D^2 - 3D + 2)\, y = 10\, e^{3x}$$

A.E. is $m^2 - 3m + 2 = 0$

i.e., $(m - 2)(m - 1) = 0$

$$m = 2, 1$$

$$\text{C.F.} = C_1 e^{2x} + C_2 e^{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} 10e^{3x} \qquad (D \to 3)$$

$$= \frac{1}{3^2 - 3 \times 3 + 2} 10e^{3x}$$

$$\text{P.I.} = \frac{10\, e^{3x}}{2}$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{x} + \frac{10\, e^{3x}}{2}.$$

3. Solve:

$(D^2 + 9)\, y = \cos 4x.$

Solution. Given equation is $(D^2 + 9)\, y = \cos 4x$

A.E. is $m^2 + 9 = 0$

i.e., $m = \pm 3i$

$$\text{C.F.} = C_1 \cos 3x + C_2 \sin 3x$$

$$\text{P.I.} = \frac{1}{D^2+9} \cos 4x \qquad (D^2 \to -4^2 = -16)$$

$$= \frac{1}{-16+9} \cos 4x = -\frac{1}{7} \cos 4x$$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 \cos 3x + C_2 \sin 3x - \frac{1}{7} \cos 4x.$$

4.Solve:

$(D^2 + D + 1)\, y = \sin 2x.$

Solution. The A.E. is

$$m^2 + m + 1 = 0$$

i.e.,
$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

Hence the C.F. is

$$\text{C.F.} = e^{-\frac{x}{2}} \left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1} \sin 2x \qquad (D^2 \to -2^2)$$

$$= \frac{1}{-2^2 + D + 1} \sin 2x$$

$$= \frac{1}{D - 3} \sin 2x$$

Multiplying and dividing by $(D + 3)$

$$= \frac{(D+3) \sin 2x}{D^2 - 9}$$

$$= \frac{(D+3) \sin 2x}{-2^2 - 9} = \frac{-1}{13} (2 \cos 2x + 3 \sin 2x)$$

$$\therefore\ y = \text{C.F.} + \text{P.I.} = e^{\frac{-x}{2}} \left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right] - \frac{1}{3}\ (2 \cos 2x + 3 \sin 2x).$$

Exercise

1. $(D^2 + 1)\, y = \sin 2x.$ $\left[\textbf{Ans. } y = C_1 \cos x + C_2 \sin x - \frac{1}{3} \sin 2x \right]$

2. $(D^2 - 4)\, y = \sin 2x + \cos 3x.$ $\left[\textbf{Ans. } y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{8} \sin 2x - \frac{1}{13} \cos 3x \right]$

3. $(D^2 + 9)\, y = \sin 3x.$ $\left[\textbf{Ans. } y = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{6} \cos 3x \right]$

4. $(D^2 + 16)\, y = \cos 4x.$ $\left[\textbf{Ans. } y = C_1 \cos 4x + C_2 \sin 4x + \frac{x}{8} \sin 4x \right]$

5. $(D^2 + 1)\, y = \sin x \sin 2x.$ $\left[\textbf{Ans. } y = C_1 \cos x + C_2 \sin x + \frac{1}{16} (4x \sin x + \cos 3x) \right]$

5. Solve:

$y'' + 3y' + 2y = 12x^2.$

Solution. We have $(D^2 + 3D + 2)\, y = 12x^2$

A.E. is $\quad m^2 + 3m + 2 = 0$

i.e., $\quad (m + 1)(m + 2) = 0$

$\Rightarrow \quad m = -1, -2$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^{-2x}$$

$$\text{P.I.} = \frac{12x^2}{D^2 + 3D + 2}$$

We need to divide for obtaining the P.I.

$$\begin{array}{r|l} & 6x^2 - 18x + 21 \\ \hline 2 + 3D + D^2 & 12x^2 \\ & 12x^2 + 36x + 12 \\ \hline & \quad - 36x - 12 \\ & \quad - 36x - 54 \\ \hline & \qquad\qquad 42 \\ & \qquad\qquad 42 \\ \hline & \qquad\qquad 0 \end{array}$$

Note:
$3D(6x^2) = 36x$
$D^2(6x^2) = 12$

Hence, P.I. $= 6x^2 - 18x + 21$

$\therefore$ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$
$$y = C_1 e^{-x} + C_2 e^{-2x} + 6x^2 - 18x + 21.$$

6. Solve:

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 2x + x^2.$$

Solution. We have $(D^2 + 2D + 1)\, y = 2x + x^2$

A.E. is $\quad m^2 + 2m + 1 = 0$

i.e., $\quad (m + 1)^2 = 0$

i.e., $\quad (m + 1)(m + 1) = 0$

$\Rightarrow \quad m = -1, -1$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{2x + x^2}{D^2 + 2D + 1} = \frac{x^2 + 2x}{1 + 2D + D^2}$$

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\ \hline
1 + 2D + D^2 & x^2 + 2x \\
 & x^2 + 4x + 2 \\ \hline
 & -2x - 2 \\
 & -2x - 4 \\ \hline
 & 2 \\
 & 2 \\ \hline
 & 0
\end{array}
$$

$\therefore$ $\quad$ P.I. $= x^2 - 2x + 2$

$\therefore$ $\quad$ $y =$ C.F. + P.I.

$= (C_1 + C_2 x)\, e^{-x} + (x^2 - 2x + 2).$

7.Solve:

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} - 3y = e^x \cos x.$$

Solution. We have

$$(D^2 + 2D - 3)\, y = e^x \cos x$$

A.E. is $\quad m^2 + 2m - 3 = 0$

i.e., $\quad (m + 3)(m - 1) = 0$

i.e., $\quad m = -3, 1$

$$\text{C.F.} = C_1 e^{-3x} + C_2 e^x$$

$$\text{P.I.} = \frac{1}{D^2 + 2D - 3} e^x \cos x$$

Taking e^x outside the operator and changing D to $D + 1$

$$= e^x \frac{1}{(D+1)^2 + 2(D+1) - 3} \cos x$$

$$= e^x \frac{1}{D^2 + 4D} \cos x \qquad (D^2 \to -1^2)$$

$$= e^x \frac{1}{-1 + 4D} \cos x$$

$$= e^x \left[\frac{\cos x}{4D - 1} \times \frac{4D + 1}{4D + 1}\right]$$

$$= e^x \left[\frac{-4 \sin x + \cos x}{16 D^2 - 1}\right] \qquad (D^2 \to -1^2)$$

$$= e^x \left[\frac{-4\sin x + \cos x}{-17}\right]$$

$$= \frac{e^x}{17}\,(4\sin x - \cos x)$$

$\therefore$ $$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{-3x} + C_2 e^x + \frac{e^x}{17}\,(4\sin x - \cos x).$$

8.Solve:

$(D^3 + 1)\,y = 5e^x x^2.$

Solution. A.E. is

$$m^3 + 1 = 0$$

i.e., $(m+1)(m^2 - m + 1) = 0$

$$(m+1) = 0,\ m^2 - m + 1 = 0$$

$$m = -1$$

$$m = \frac{1 \pm \sqrt{3}i}{2}$$

$$\text{C.F.} = C_1 e^{-x} + e^{\frac{x}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1}\,5e^x x^2$$

Taking e^x outside the operator and changing D to $D + 1$

$$= e^x \frac{1}{(D+1)^3 + 1} \cdot 5x^2$$

$$= e^x \frac{5x^2}{D^3 + 3D^2 + 3D + 2}$$

$$= \frac{5e^x}{2}\left[\frac{2x^2}{2+3D+3D^2+D^3}\right]$$

(For a convenient division we have multiplied and divided by 2)

$$\begin{array}{r|l}
 & x^2 - 3x + \frac{3}{2} \\
\hline
2 + 3D + 3D^2 + D^3 & 2x^2 \\
 & 2x^2 + 6x + 6 \\
\hline
 & \quad -6x - 6 \\
 & \quad -6x - 9 \\
\hline
 & \qquad\quad 3 \\
 & \qquad\quad 3 \\
\hline
 & \qquad\quad 0
\end{array}$$

$$\therefore \qquad \text{P.I.} = \left(x^2 - 3x + \frac{3}{2}\right)\cdot\frac{5e^x}{2}$$

$$= \frac{5e^x}{4}\ (2x^2 - 6x + 3)$$

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1\, e^{-x} + e^{\frac{x}{2}}\left\{C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right\} + \frac{5e^x}{4}(2x^2 - 6x + 3).$$

9.Solve:

$$\frac{d^2y}{dx^2} + 4y = x \sin x.$$

Solution. We have

$$(D^2 + 4)\, y = x \sin x$$

A.E. is
$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i$$

$$\text{C.F.} = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\, x \sin x$$

Let us use $$\frac{xV}{f(D)} = \left[x - \frac{f'(D)}{f(D)}\right]\frac{V}{f(D)}$$

$$\frac{x \sin x}{D^2+4} = \left[x - \frac{2D}{D^2+4}\right]\frac{\sin x}{D^2+4} \qquad (D^2 \to -1^2)$$

$$= \frac{x \sin x}{D^2+4} - \frac{2\,D(\sin x)}{(D^2+4)^2} \qquad (D^2 \to -1^2)$$

$$= \frac{x \sin x}{3} - \frac{2 \cos x}{3^2}$$

$$= \frac{x \sin x}{3} - \frac{2 \cos x}{9}$$

$$\text{P.I.} = \frac{1}{9}(3x \sin x - 2 \cos x)$$

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 \cos 2x + C_2 \sin 2x + \frac{1}{9}(3x \sin x - 2 \cos x).$$

10 .Solve:

$(D^2 + 2D + 1)\, y = x \cos x.$

Solution. A.E. is

$$m^2 + 2m + 1 = 0$$

i.e., $$(m+1)^2 = 0$$

$$m = -1, -1$$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{x \cos x}{D^2 + 2D + 1}.$$

$$\text{P.I.}_1 = \frac{x}{2}\sin x$$

$$\text{P.I.}_2 = \frac{(2D+2)\cos x}{(D^2+2D+1)^2} \qquad (D^2 \to -1^2)$$

$$= \frac{-2\sin x + 2\cos x}{(2D)^2}$$

$$= \frac{-2\sin x + 2\cos x}{4D^2} \qquad (D^2 = -1^2)$$

$$= \frac{2\sin x - 2\cos x}{4}$$

$$= \frac{1}{2}(\sin x - \cos x)$$

$$\text{P.I.} = \frac{1}{2}x\sin x - \frac{1}{2}(\sin x - \cos x)$$

$$= \frac{1}{2}(x\sin x - \sin x + \cos x)$$

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (C_1 + C_2 x)e^{-x} + \frac{1}{2}(x\sin x - \sin x + \cos x).$$

Exercise

1. $(D^2 + 9)y = x\cos x.$ $\left[\textbf{Ans. } y = C_1\cos 3x + C_2\sin 3x + \frac{1}{32}(4x\cos x + \sin x)\right]$

2. $(D^2 - 2D + 1)y = x\sin x.$ $\left[\textbf{Ans. } y = (C_1 + C_2 x)e^{-x} + \frac{1}{2}(\sin x + \cos x - 1)\right]$

3. $(D^2 - 1)y = x\sin 3x.$ $\left[\textbf{Ans. } y = C_1 e^{x} + C_2 e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x)\right]$

4. $(D^2 - 3D + 2)y = x\cos 2x.$

$\left[\textbf{Ans. } y = C_1 e^{x} + C_2 e^{2x} - \frac{1}{20}x(3\sin 2x + \cos 2x) - \frac{1}{200}(7\sin 2x + 24\cos 2x)\right]$

5. $\frac{d^2y}{dx^2} + a^2 y = x\cos ax.$ $\left[\textbf{Ans. } y = C_1\cos ax + C_2\sin ax + \frac{1}{4a^2}(ax^2\sin ax + x\cos ax)\right]$

2.3 Exact Solutions and Integrating Factors

Exact differential equations are a subset of ordinary differential equations where a differential equation is said to be exact if it can be expressed in the form:

M (x ,y) dx + N (x,y) dy = 0

Where M and N are functions of two variables x and y, defined on a simply connected region D in the plane. The equation is called "exact" if there exists a function F(x,y) such that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

11 .Solve:

$$\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$$

Sol: Hence $M = 1+e^{\frac{x}{y}}$ & $N = e^{\frac{x}{y}}(1-\frac{x}{y})$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \& \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + (1-\frac{x}{y})e^{\frac{x}{y}}(\frac{1}{y})$$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \& \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ equation is exact

General solution is

$$\int M dx \quad + \int N dy \ = c.$$

(y constant) (terms free from x)

$$\int (1 + e^{\frac{x}{y}})\, dx \quad + \int 0\, dy \ = c.$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\Rightarrow x + y\, e^{\frac{x}{y}} = C$$

12. Solve:

$$(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0$$

Solution. Here, $M = 5x^4 + 3x^2y^2 - 2xy^3$, $N = 2x^3y - 3x^2y^2 - 5y^4$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2, \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Now $\int M\, dx + \int (\text{terms of } N \text{ is not containing } x)\, dy = C$ (y constant)

$$\int \left(5x^4 + 3x^2y^2 - 2xy^3\right) dx + \int -5y^4 dy = C$$

$$\Rightarrow \quad x^5 + x^3y^2 - x^2y^3 - y^5 = C$$ **Ans.**

2.4 Finding Solutions When a Part of the Complementary Function (C.F.) is Known

Let y = uv be the complete solution of DE y”+Py’+Qy=R then

$$y' = uv' + u'v \qquad y'' = uv'' + 2u'v' + u''v$$

y'' + Py' + Qy = R

$$uv'' + 2u'v' + u''v + P(uv' + u'v) + Q(uv) = R$$

$$(u)v'' + (2u' + Pu)v' + (u'' + Pu' + Qu)v = R$$

(a) Choose u such that u''+Pu'+Qu=0 i.e u is the solution of given DE.

(b) Choose u such that 2u'+Pu=0.T

$$v''(u) + v'(2u' + Pu) = R \qquad v'' + \left(\frac{2}{u}u' + P\right)v' = \frac{R}{u}$$

$$x^2 y'' - \left(x^2 + 2x\right) y' + (x+2) y = x^3 e^x$$

Since y = x satisfies the differential equation. We can choose u = x

$$y'' - \left(\frac{x^2 + 2x}{x^2}\right) y' + \left(\frac{x+2}{x^2}\right) y = xe^x$$

$$v'' + \left(\frac{2}{x}(1) - \frac{x^2 + 2x}{x^2}\right) v' = \frac{xe^x}{x}$$

$$v'' - v' = e^x$$

$$Put\, v' = w \;\; and \;\; v'' = w'$$

$$\frac{dw}{dx} - w = e^x \qquad I.F. = e^{-x}$$

$$we^{-x} = x + c_1 \qquad \frac{dv}{dx} = xe^x + c_1 e^x$$

$$v = (x-1)e^x + c_1 e^x + c_2$$

$$\mathrm{y} = \mathrm{uv} = (\mathrm{x} - 1 + \mathrm{c}_1)\mathrm{xe}^{\mathrm{x}} + \mathrm{c}_2 \mathrm{x}$$

2.5 Summary

Second-order ordinary differential equations (ODEs) are mathematical expressions that involve the second derivative of an unknown function with respect to a single independent variable.

2.6 Keywords

1. Second Order Differential Equations
2. ODEs
3. Exact ODEs
4. Homogeneous Equations
5. Non-Homogeneous Equations

2.7 Self-Assessment questions

1. Define a second-order ordinary differential equation (ODE) and explain its significance in mathematical modeling.

2. What are the differences between a linear and a nonlinear second-order ode? Provide examples of each.

3. How do you determine whether a second-order ODE is homogeneous or non-homogeneous?

4. Discuss the significance of initial value problems (IVPs) and boundary value problems (BVPs) in the context of second-order ODEs.

5. Describe the method of undetermined coefficients and when it is applicable in solving non homogeneous second-order ODEs.

2.8 Case Study

Rhythmic Mass, spring, and Damper Mechanism

Imagine you have a mass (m), a spring (k), and a damper (c) with a damping coefficient. This is a mass-spring-damper system. An extemal force (F (t)) causes the mass to shift from its equilibrium position by a distance (x (t)).

1. Determine the differential equation controlling the mass's motion by analyzing its motion.
2. Determine the specific solution when (F (t)), using a variety of techniques such variable

2.9 References

- Kristensson, G. (2010). Second Order Differential Equations: Special Functions and Their Classification. Germany: Springer New York.
- Keskin, A. Ü. (2018). Ordinary Differential Equations for Engineers: Problems with MATLAB Solutions. Germany: Springer International Publishing.

UNIT - 3

Variables in ODEs

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure:

3.1 Change of Variables in ODEs

Change of variables is a technique often used in solving ordinary differential equations (ODEs) to simplify the problem or make it more amenable to solution. The idea is to substitute one set of variables with another set that might simplify the equation or make it easier to integrate.

Here's a general outline of how change of variables works in the context of ODEs:

1. **Identify the ODE**: Start with the ordinary differential equation you want to solve. This could be a first-order or higher-order equation.
2. **Choose a New Variable**: Select a new variable or set of variables to replace the original ones. The choice of new variables depends on the structure of the ODE and the desired simplification.
3. **Express the ODE in Terms of New Variables**: Substitute the new variables into the original ODE, replacing all occurrences of the original variables.

3.2 Change of Independent Variables

Change of independent variables, also known as a change of variables or transformation of variables, involves altering the independent variable in a differential equation. This technique is often used to simplify the equation or make it more tractable for analysis. Change of independent variables is particularly useful when dealing with differential equations that involve complex functions or where the structure of the equation can be simplified by a suitable transformation. It can also help in identifying symmetries or patterns in the equations that might not be apparent in their original form.

1. Solve:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4} y = 0, \ x \neq 0,$$

Solution : Comparing the given equation with Eqn. (4), we get

$$p = \frac{2}{x},\ q = \frac{a^2}{x^4} \text{ and } r = 0.$$

Let us find z so that

$$q_1 = \frac{q}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = c_1 \text{ (say), where } c_1 \neq 0.$$

$$\Rightarrow c_1 \left(\frac{dz}{dx}\right)^2 = q = \frac{a^2}{x^4}$$

Solving, we get

$$z = -\frac{a}{\sqrt{c_1}x}$$

Using Eq.(8) $p_1=0$, $q_1=$ constant so

$$\frac{d^2y}{dz^2} + c_1 y = 0,$$

$$\Rightarrow y = A\cos\sqrt{c_1}z + B\sin\sqrt{c_1}z$$

Hence the solution of the given differential equation is

$$y = A\cos\left(\frac{a}{x}\right) - B\sin\left(\frac{a}{x}\right).$$

2. Solve :

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = c^2 y \text{ for } |x| < 1 \text{ and c being a constant.}$$

Solution : The given differential equation can be written as

$$\frac{d^2y}{dx^2} - \frac{x}{1 - x^2}\frac{dy}{dx} - \frac{c^2}{1 - x^2}y = 0$$

Let us now change the independent variable x to z, so that

$$p_1 = 0.$$

$$\Rightarrow \frac{dz}{dx} = e^{-\int \frac{-x}{1-x^2}dx} = e^{-\frac{1}{2}\ln(1-x^2)} = (1 - x^2)^{-1/2}$$

and integrating, we get

$$z = \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$$

With this value of z, we have

$$q_1 = \frac{q}{\left(\frac{dz}{dx}\right)^2} = \frac{-c^2}{(1 - x^2)}(1 - x^2) = -c^2, \text{ which is a constant.}$$

∴ Eqn. (11) takes the form

$$\frac{d^2y}{dx^2} - c^2y = 0$$

$$\Rightarrow y = Ae^{cz} + Be^{-cz}$$
$$= Ae^{c\sin^{-1}x} + Be^{-c\sin^{-1}x},$$

which is the required solution.

3.3 Change of dependent Variables

Change of dependent variable is a technique used to simplify the form of a differential equation by introducing a new dependent variable. This transformation can convert a nonlinear or higher-order equation into a simpler form, often linear or first-order. The process involves defining a new function in terms of the original dependent variable, substituting it into the equation, and manipulating the resulting expression to achieve a desired form. This technique is particularly useful for solving differential equations analytically or numerically, enhancing understanding and facilitating the application of various solution methods

3.Solve :

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x^{m+1}$$

Solution : Through inspection, we find that y = x is a solution of the homogeneous equation

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$$

Thus z = v(x) becomes z = x and the substitution y = vz transforms to y = vx.

On substituting y = vx in Eqn. (24), we get

$$\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - x^2\left(x\frac{dv}{dx} + v\right) + x(xv) = x^{m+1}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{2}{x} - x^2\right)\frac{dv}{dx} = x^m$$

The above differential equation is linear in $\frac{dv}{dx}$ and its integrating factor is

$$e^{\int\left(\frac{2}{x} - x^2\right)dx} = e^{2\ln x - \frac{x^2}{3}} = x^2e^{-\frac{x^3}{3}}$$

Hence

$$x^2 e^{-\frac{x^3}{3}}\frac{dv}{dx} = A + \int x^{m+2} e^{-\frac{x^3}{3}} dx$$

$$\Rightarrow \frac{dv}{dx} = \frac{A}{x^2} e^{\frac{x^3}{3}} + \frac{1}{x^2} e^{\frac{x^3}{3}}\left(\int x^{m+2} e^{-\frac{x^3}{3}} dx\right)$$

Integrating the above equations, we get

$$v = B + \int \frac{A}{x^2} e^{\frac{x^3}{3}} dx + \int \frac{1}{x^2} e^{\frac{x^3}{3}}\left(\int x^{m+2} e^{-\frac{x^3}{3}} dx\right) dx,$$

so that the complete solution of Eqn. (24) is

$$y = vx = Bx + x\int \frac{A}{x^2} e^{\frac{x^3}{3}} dx + x\int \frac{1}{x^2} e^{\frac{x^3}{3}}\left(\int x^{m+2} e^{-\frac{x^3}{3}} dx\right) dx,$$

where A and B are arbitrary constants.

$$y = Bx + x\int \frac{A}{x^2} e^{\frac{x^3}{3}} dx - x\int \frac{1}{x^2} dx.$$

$$= Bx + x\int \frac{A}{x^2} e^{\frac{x^3}{3}} dx + 1$$

where n is positive integer, then

$$\int x^{m+2} e^{-\frac{x^3}{3}} dx = \int x^{3n+2} e^{-\frac{x^3}{3}} dx. \left(\text{Let } -\frac{x^3}{3} = t, \therefore -x^2\,dx = dt\right)$$

$$= -3^n \int t^n e^t\, dt.$$

Integrating n times, we get

$$\int x^{m+2} e^{-\frac{x^3}{3}} dx = e^t\, 3^n\, [t^n + (-1)^1 nt^{n-1} + (-1)^2 n(n-1)\, t^{n-2}$$
$$+ (-1)^3 n(n-1)(n-2)\, t^{n-3} + \ldots + (-1)^{n-1} n(n-1) \ldots 2t$$
$$+ (-1)^n n(n-1) \ldots 2.1]$$

$$= -3^n e^{-\frac{x^3}{3}} \left[\left(-\frac{x^3}{3}\right)^n + (-1)\, n\left(-\frac{x^3}{3}\right)^{n-1} + (-1)^2 n(n-1)\left(-\frac{x^3}{3}\right)^{n-2}\right.$$
$$\left. + \ldots + (-1)^{n-1} n(n-1) \ldots 2\left(-\frac{x^3}{3}\right) + (-1)^n n(n-1) \ldots 2.1\right]$$

4.Solve :

$$\frac{d^2y}{dx^2} - \frac{1}{x^{1/2}} \frac{dy}{dx} + \frac{y}{4x^2}(-8 + x^{1/2} + x) = 0$$

Solution : On comparing the given equation with Eqn. (4), we find that

$$p = -\frac{1}{x^{1/2}}, \ q = \frac{1}{4x^2}(-8 + x^{1/2} + x), \ r = 0.$$

If we substitute $y = vz$ and make use of Eqn. (20), we get

$$z = e^{-\frac{1}{2}\int -\frac{1}{x^{1/2}}dx} = e^{x^{1/2}}.$$

Also, from Eqn. (23), $q_1 = q - \frac{1}{2}\frac{dp}{dx} - \frac{1}{4}p^2$

$$= \frac{1}{4x^2}(-8 + x^{1/2} + x) - \frac{1}{4x^{3/2}} - \frac{1}{4}\frac{1}{x}$$

$$= -\frac{2}{x^2}.$$

and Eqn. (22), giving v, reduces to

$$\frac{d^2v}{dx^2} - \frac{2}{x^2}v = 0.$$

Above equation is a particular case of Euler's equation and its solution is

$$v = \frac{A}{x} + Bx^2.$$

Thus the general solution of Eqn. (26) can be expressed as

$$y = vz = \left(\frac{A}{x} + Bx^2\right)e^{\sqrt{x}}.$$

3.4 Summary

Ordinary Differential Equations (ODEs) are mathematical equations involving derivatives of a function with respect to a single independent variable. They come in different orders and types (linear, nonlinear, homogeneous, non-homogeneous). ODEs find applications in modeling dynamic systems across physics, engineering, biology, and economics. Solutions can be obtained analytically or numerically, and initial value problems specify conditions at a starting point.

3.5 Keywords

1. Transformation
2. Substitution
3. Standard forms

4. Simplification

5. Variable change

3.6 Self Assessment Questions

1. What is the primary purpose of employing change of variables in ODEs?
2. Describe at least two techniques for changing variables in ODEs.
3. How do you identify appropriate transformations for a given ODE?
4. Explain the geometric interpretation of change of variables in the context of systems of equations
5. How is change of variables in ODEs connected to concepts from linear algebra?
6. Provide an example of an application of change of variables in a scientific or engineering context.

3.7 Case Study

The Use of Heat Transfer in Engineering Design

As an engineer, you can be assigned the responsibility of creating a cooling system for a high-performing electrical gadget, such a high-power laser or a CPU. To avoid overheating and guarantee optimum Performance and lifespan, it is essential to comprehend how heat drains from the device during the design phase.

Question: Your goal is to optimize the cooling system design and simplify the analysis by modeling the heat transfer process with differential equations and change of variables.

3.8 References

- Kristensson, G. (2010). Second Order Differential Equations: Special Functions and Their Classification. Germany: Springer New York.
- Keskin, A. Ü. (2018). Ordinary Differential Equations for Engineers: Problems with MATLAB Solutions. Germany: Springer International Publishing.

UNIT - 4

Parameters

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure

4.1 Variation of Parameters

Variation of parameters is a method used to find a particular solution of a non-homogeneous linear differential equation, especially second-order equations. It extends the principle of superposition by assuming the particular solution is a linear combination of functions whose coefficients vary with the independent variable. By substituting this assumption into the differential equation and solving for the coefficients using the method of undetermined coefficients, one can find the particular solution. This technique is valuable for solving non-homogeneous linear differential equations with variable coefficients

4.2 Derivation and Application of Variation of Parameters Method

1. Solve:

$$2y'' + 18y = 6\tan 3t$$

$$y_c(t) = c_1\cos(3t) + c_2\sin(3t)$$

$$y_1(t) = \cos(3t) \qquad y_2(t) = \sin(3t)$$

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3$$

$$\begin{aligned} Y_P(t) &= -\cos(3t)\int \frac{3\sin(3t)\tan(3t)}{3}\,dt + \sin(3t)\int \frac{3\cos(3t)\tan(3t)}{3}\,dt \\ &= -\cos(3t)\int \frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt \\ &= -\cos(3t)\int \frac{1-\cos^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int \sin(3t)\,dt \\ &= -\cos(3t)\int \sec(3t) - \cos(3t)\,dt + \sin(3t)\int \sin(3t)\,dt \end{aligned}$$

$$= -\frac{\cos(3t)}{3}(\ln|\sec(3t)+\tan(3t)| - \sin(3t)) + \frac{\sin(3t)}{3}(-\cos(3t))$$
$$= -\frac{\cos(3t)}{3}\ln|\sec(3t)+\tan(3t)|$$

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(3t)}{3}\ln|\sec(3t)+\tan(3t)|$$

4.3 Solving Second Order ODEs using Variation of Parameters

2. Solve :

Given

$y_1(x) = e^x$ and $y_2(x) = e^{2x}$

$$y'' - 3y' + 2y = 0.$$

Their Wronskian is:

$$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x(2e^{2x}) - e^{2x}(e^x) = e^{3x} \neq 0 \text{ for all } x \in (-\infty, \infty).$$

From Example 2, the functions $y_1(x) = x^2$ and $y_2(x) = 5x^2$ are solutions of

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0.$$

Their Wronskian is:

$$W(x) = \begin{vmatrix} x^2 & 5x^2 \\ 2x & 10x \end{vmatrix} = x^2(10x) - 2x(5x^2) = 10x^2 - 10x^2 \equiv 0.$$

4.4 Summary

Variation of parameters is a technique used to solve non-homogeneous linear differential equations. The method extends the principle of superposition by assuming the particular solution as a linear combination of functions with varying coefficients.

4.5 Keywords

1. Complementary Function
2. Particular Solution
3. Linear Independence
4. Linear Combination
5. Characteristic Equatio
6. Roots of Equations

4.6 Self Assessment Questions

1. What type of differential equations does the variation of parameters method apply to?
2. What is the first step in applying the variation of parameters method to solve a non homogeneous linear differential equation?
3. How is the complementary function related to the variation of parameters method?
4. Describe the process of guessing a particular solution in the variation of parameters method.
5. What does it mean for functions to be linearly independent?

4.7 Case Study

Improving Customer Satisfaction in a Restaurant, Imagine you're the manager of a restaurant facing a decline in customer satisfaction despite serving delicious food. You've identified that long wait times and inconsistent service are the main issues.

Question: Your restaurant's current operational model relies heavily on fixed procedures and limited flexibility in handling customer needs. This rigidity leads to dissatisfaction when customers' preferences or needs aren't met promptly.

4.8 References

1. William. E.(2017). Elementary Differential Equations and Boundary Value Problems. John Wiley & Sons.
2. Erwin Kreyszig (2020). Advanced Engineering Mathematics. John Wiley & Sons

UNIT 5

Series Solutions

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure

5.1 Solution in Series

A Series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots ,$$

1.Solve : A Power series for

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots .$$

Since

$\frac{d}{dx}\{1/(1-x)\} = \frac{1}{(1-x)^2}$,

We obtain a power series for

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots .$$

Since $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$, integrate the series for $\frac{1}{1+x^2}$ termwise to obtain

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots .$$

Shifting the summation index :

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{i=0}^{\infty} a_i(x-x_0)^i .$$

Example:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k .$$

$$x^3 \sum^{\infty} n^2(n-2)a_n x^n = \sum^{\infty}(n-3)^2(n-5)a_{n-3}x^n .$$

Definition: (Analytic function)

A function f is said to be analytic at x_0 if it has a power series representation

$\sum_{n=0}^{\infty} a_n(x-x_0)^n$

in an neighborhood about x_0,and has a positive radius of convergence.

Example: Some analytic functions and their representations:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n, \quad x > 0.$$

5.2 Power Series Solutions of ODEs

Power series solutions are a powerful technique for solving ordinary differential equations (ODEs), especially when an exact solution cannot be found using other methods. The method involves assuming that the solution to the ODE can be represented by a power series, then substituting this series into the ODE and solving for the coefficients of the series.

Standard form

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad a_2(x) \neq 0.$$

Writing in the standard form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where $p(x) := a_1(x)/a_2(x)$ and $q(x) := a_0(x)/a_2(x)$.

2. Solve: Find all singular points of

$$xy''(x) + x(x-1)^{-1}y'(x) + (\sin x)y = 0, \quad x > 0$$

$$p(x) = \frac{1}{(1-x)}, \quad q(x) = \frac{\sin x}{x}.$$

$$q(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

Note that p(x) is analytic except at x=1 . q(x) is analytic everywhere as it has power series Hence x=1 is only singular point.

5.3 Radius of Convergence and Interval of Convergence

The radius of convergence and interval of convergence are crucial concepts in the study of power series, which are infinite series of the form:

$$\sum_{n=0}^{\infty} a_n x^n$$

The radius of convergence, denoted by R, is a non-negative real number or +∞ to -∞, associated with a power series. It represents the radius of the largest interval centered at the series' center for which the series converges. Mathematically, it's defined as:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

where a_n are the coefficients of the series. The power series converges absolutely within its interval of convergence and diverges outside of it. At the endpoints of the interval of convergence, the convergence behavior may vary. The series might converge or diverge, and further investigation, such as the ratio test or other convergence tests, may be necessary to determine convergence at the endpoints. Understanding the radius and interval of convergence is essential for applications such as solving differential equations using power series methods and for ensuring the validity of approximations using power series expansions.

3. Solve :

Find radius of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^n . n(x+3)^n}{4^n}$$

Solution:

$\sum_{n=1}^{\infty} \frac{(-1)^n . n(x+3)^n}{4^n}$

Let us find the radius of convergence using the ratio test.

Thus, a_n =

$\frac{(-1)^n . n(x+3)^n}{4^n}$

and a_{n+1} =

$\frac{(-1)^{n+1} . (n+1)(x+3)^{n+1}}{4^{n+1}}$

Now, we have to take the limit as n → ∞ for the absolute ratio of a_{n+1}/a_n.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \left| \frac{(-1)^{n+1} . (n+1)(x+3)^{n+1}}{4^{n+1}} \times \frac{4^n}{(-1)^n . n(x+3)^n} \right|$$

This can be simplified as:

$$= \lim_{n\to\infty} \left| \frac{(-1) . (n+1)(x+3)}{4n} \right|$$

Now, taking the term that contains x outside limit, we get;

$$|x+3| \lim_{n\to\infty} \frac{(n+1)}{4n}$$

= (¼) |x + 3|

This is of the form N.(x – a).

Therefore, the radius of convergence = R = 1/N = 1/(¼) = 4

5.4 Method of Differentiation and Cauchy-Euler Equations

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_0 y = \phi(x)$$

Where $a_0, a_1, a_2, \ldots$ are constants, is called a homogeneous equation.

Put $x = e^z$, $z = \log_e x$, $\frac{d}{dz} = D$

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz} \quad \Rightarrow \quad x\frac{dy}{dx}=\frac{dy}{dz} \quad \Rightarrow \quad x\frac{dy}{dx}=Dy$$

Again, $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x}=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)=\frac{1}{x^2}(D^2-D)y$$

$$x^2\frac{d^2y}{dx^2}=(D^2-D)y$$

$\Rightarrow$ $$\boxed{x^2\frac{d^2y}{dx^2}=D(D-1)y}$$

Similarly. $$\boxed{x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)\,y}$$

4. Solve :

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=\cos(\log x)+x\sin(\log x).$$

Solution:

We have, $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4y=\cos(\log x)+x\sin(\log x).$

Putting $x=e^z \Rightarrow z=\log x, D=\frac{d}{dz}$ and $x^2\frac{d^2y}{dx^2}=D(D-1)y, x\frac{dy}{dx}=Dy$ in (1), we get

$$[D(D-1)-D+4]y=\cos z+e^z\sin z$$

i.e. $$(D^2-2D+4)y=\cos z+e^z\sin z$$

A.E. is $m^2-2m+4=0 \Rightarrow \qquad m=\frac{-(-2)\pm\sqrt{4-16}}{2}$

$$m = 1 \pm \sqrt{3}\, i$$

$$\text{C.F.} \qquad = e^{z}[C_1 \cos\sqrt{3}\, z + C_2 \sin\sqrt{3}\, z]$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4}(\cos z + e^{z} \sin z)$$

$$= \frac{1}{D^2 - 2D + 4}\cos z + \frac{1}{D^2 - 2D + 4} e^{z} \sin z$$

$$= \frac{1}{-1 - 2D + 4}\cos z + e^{z} \frac{1}{(D+1)^2 - 2(D+1) + 4} \sin z$$

$$y = C.F. + P.I.$$

$$y = e^{z}[C_1 \cos\sqrt{3}\, z + C_2 \sin\sqrt{3}\, z] + \frac{1}{13}(3\cos z - 2\sin z) + \frac{1}{2} e^{z} \sin z$$

Replacing $z = \log x$ and $e^{z} = x$ in (4), we get

$$y = x[C_1 \cos\sqrt{3}(\log x) + C_2 \sin\sqrt{3}(\log x)]$$

$$+ \frac{3}{13}\cos(\log x) - \frac{2}{13}\sin(\log x) + \frac{1}{2} x \sin(\log x)$$

5.5 Summary

Series solutions are a method used in mathematics to find solutions to certain types of differential equations. These equations often arise in physics, engineering, and other scientific disciplines. Differential equations that cannot be solved by standard methods often can be approached through series solutions. These equations typically involve functions and their derivatives. The idea behind series solutions is to express the solution to the differential equation as a power series, where each term is a polynomial multiplied by a coefficient.

5.6 Keywords

1. Power Series
2. Coefficients
3. Recurrence Relations

4. Convergence
5. Singular Points
6. Frobenius Method
7. Taylor Series
8. Maclaurin Series

5.7 Self Assessment questions

1. What is the purpose of using series solutions in differential equations?

2. How do you express a solution as a power series?

3. What is a recurrence relation in the context of series solutions?

4. Explain the concept of convergence in series solutions.

5. What are some common techniques for solving recurrence relations?

5.8 Case Study

An analysis of electrical circuits:

When conventional techniques become unfeasible owing to circuit complexity or nonlinearity, series solutions can be utilized in electrical engineering to examine intricate circuits. **Question:** Learn about the behavior and functionality of a complicated electrical circuit by analyzing it.

5.9 References

1. Evans, L. C. (2010). Partial Differential Equations (2nd ed.). American Mathematical Society.
2. Strauss, W. A. (2008). Partial Differential Equations: An Introduction. John Wiley & Sons.

UNIT - 6

Method of Undetermined Coefficients

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure

6.1 Partial Differential Equations (PDEs) of First Order

Partial differential equations (PDEs) of first order are mathematical equations that involve partial derivatives of a function of several independent variables. They typically describe physical phenomena such as heat conduction, fluid dynamics, and electromagnetism. A first-order PDE involves only first-order partial derivatives of the unknown function.

for z = f (x,y)

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

6.2 Lagrange's Method and Standard Forms

An equation of the form Pp+Qq=R is said to be Lagrange's type of P.D.E.

1.Solve: a(p+q)=z

Solution :

Given $ap + aq = z$

A.E

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{1}$$

From first and second

$$dx - dv = 0$$

$$x - y = c_1$$

Taking last two

dy-a dz=0

On integrating

$y - az = c_2$

$\phi(x-y, y-az) = 0$

2.Solve: 2p+3q=1

Solution:

2p+3q=1

A.E

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1}$$

$$3dx - 2dy = 0$$

$$3x - 2y = c_1$$

$$dy - 3dz = 0$$

$y-3z=c_2$

$$\phi(3x - 2y, y - 3z) = 0$$

3.Solve : xzp+yzp=xy

Solution:

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

$$\frac{dx}{x} - \frac{dy}{y} = 0$$

$$\log x - \log y = \log c_1 \quad or \quad x/y = c_1$$

$$x = c_1 y.$$

$$\frac{dy}{yz} = \frac{dz}{c_1 y^2} \quad \text{or} \quad c_1 y dy - z dz = 0$$

$$\tfrac{1}{2} c_1 y^2 - \tfrac{1}{2} z^2 = \tfrac{1}{2} c_2$$

$$c_1 y^2 - z^2 = c_2$$

$$xy - z^2 = c_2$$

$$\phi(xy - z^2, x/y) = 0$$

4. Solve: (y+z) p + (z+x) q = (x+y)

Solution: A.E are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Using $(1,-1,0)$ and $(1,0,-1)$

$$\frac{dx-dy}{y+z-z-x} = \frac{dx-dz}{y+z-x-y}$$

$$\frac{dx-dy}{y-x} = \frac{dx-dz}{z-x}$$

Integrating we get

or $$\int \frac{dx-dy}{y-x} = \int \frac{dx-dz}{z-x}$$

$$\log(x-y) = \log(x-z) + \log c_1$$

or $$\frac{x-y}{x-z} = c_1$$

Also using multipliers $(1,-1,0)$ and $(1,1,1)$, each fraction of (1.53) is equal to

or $$\frac{dx-dy}{y+z-z-x} = \frac{dx+dy+dz}{2(x+y+z)}$$

or $$2\frac{dx-dy}{x-y} = -\frac{dx+dy+dz}{(x+y+z)}$$

Integrating we get

$$2\log(x-y) = -\log(x+y+z) + \log c_2$$

or $$(x-y)^2(x+y+z) = c_2$$

$f(c_1, c_2) = 0$

$$f\left[\frac{x-y}{x-z},\ (x-y)^2(x+y+z)\right] = 0$$

5.Solve :

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx\ .$$

Solution: $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

The subsidiary auxiliary equations are

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{(xy+zx)} = \frac{dz}{xy-zx}$$

From 2nd and 3rd fraction we get

$$\frac{dy}{xy+zx} = \frac{dz}{xy-zx}$$

or $$\frac{dy}{dz} = \frac{y+z}{y-z}$$

It is a homogeneous equation therefore we put $y = vz$

So that $$\frac{dy}{dz} = v + z\frac{dv}{dz}$$

$$v+z\,\frac{dv}{dz}=\frac{(vz+z)}{(vz-z)}$$

$$\Rightarrow \quad z\,\frac{dv}{dz}=\frac{v+1}{v-1}-v$$

$$\Rightarrow \quad z\,\frac{dv}{dz}=\frac{v+1-v^2+v}{v-1}$$

$$\Rightarrow \quad z\,\frac{dv}{dz}=\frac{-v^2+2v+1}{v-1}$$

$$\Rightarrow \quad \frac{(v-1)dv}{v^2-2v-1}=-\frac{dz}{z}$$

$$\Rightarrow \quad -\frac{(v-1)dv}{v^2-2v-1}=\frac{dz}{z}$$

Integrating we get

$$\Rightarrow \qquad -\frac{1}{2}\int \frac{2(v-1)dv}{v^2-2v-1}=\int \frac{dz}{z}$$

$$\Rightarrow \qquad -\frac{1}{2}\log\left(v^2-2v-1\right)=\log z+\log c_1$$

$$\Rightarrow \qquad \left(v^2-2v-1\right)^{-\frac{1}{2}}=zc_1$$

$$\Rightarrow \qquad c_1=\frac{1}{\left(z^2v^2-2z^2v-z^2\right)^{\frac{1}{2}}}$$

$$\Rightarrow \qquad c_1=\frac{1}{\left(z^2\frac{y^2}{z^2}-2z^2\frac{y}{z}-z^2\right)^{\frac{1}{2}}}$$

$$\frac{xdx+ydy+zdz}{xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2}$$

$$\frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow \qquad xdx+ydy+zdz=0$$

Integrating we get

$$\Rightarrow \qquad \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_2}{2}$$

$$\Rightarrow \qquad x^2+y^2+z^2=c_2$$

The general solution is given by

$$f(c_1,c_2) = 0$$

or

$$f\left(\frac{1}{\left(y^2-2zy-z^2\right)^{\frac{1}{2}}},x^2+y^2+z^2\right) = 0$$

or $xdx = zdz$

integrating

or $\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$

or $c_2 = x^2 - z^2$

The general solution is given by

$$f(c_1, c_2) = 0$$

$$f(x^3 - y^3, x^2 - z^2) = 0$$

6.3 Charpit's Method

Charpit's method is another powerful technique for solving first-order partial differential equations (PDEs). It's particularly useful for solving quasi-linear partial differential equations of the form:

$$F(x, y, z, p, q) = 0$$

Where $z = z(x, y)$ is the unknown function, and $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Consider the equation

$$F(x, y, z, p, q) = 0 \qquad \ldots(1)$$

Since z depend on x and y, we have,

$$dZ = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = Pdx + Qdy \qquad \ldots(2)$$

Now, if we can find another relation between x, y, z, p, q such that,

$$f(x, y, z, p, q) = 0 \qquad \ldots(3)$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\cdot\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial x}=0 \qquad ...(4)$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x}=0 \qquad ...(5)$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial y}=0 \qquad ...(6)$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y}=0 \qquad ...(7)$$

Eliminating $\frac{\partial p}{\partial x}$ from equation (4) and (5), and $\frac{\partial p}{\partial y}$ from equation (6) and (7), we get

$$\left(\frac{\partial F}{\partial x}\cdot\frac{\partial f}{\partial p}-\frac{\partial f}{\partial x}\cdot\frac{\partial F}{\partial p}\right)+\left(\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial p}-\frac{\partial f}{\partial z}\cdot\frac{\partial F}{\partial p}\right)p+\left(\frac{\partial F}{\partial q}\cdot\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}\cdot\frac{\partial F}{\partial p}\right)\frac{\partial q}{\partial x}=0$$

$$\left(\frac{\partial F}{\partial y}\cdot\frac{\partial f}{\partial q}-\frac{\partial f}{\partial y}\cdot\frac{\partial F}{\partial q}\right)+\left(\frac{\partial F}{\partial z}\cdot\frac{\partial f}{\partial q}-\frac{\partial f}{\partial z}\cdot\frac{\partial F}{\partial q}\right)q+\left(\frac{\partial F}{\partial p}\cdot\frac{\partial f}{\partial q}-\frac{\partial f}{\partial p}\cdot\frac{\partial F}{\partial q}\right)\frac{\partial p}{\partial y}=0$$

Adding these two equations and using

$$\frac{\partial q}{\partial x}=\frac{\partial^2 y}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

$$\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial z}+\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial p}+\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial q}=0 \qquad ...(8)$$

$$\frac{dx}{\frac{-\partial F}{\partial p}}=\frac{dy}{\frac{-\partial F}{\partial q}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{df}{0}$$

6.Solve:

$(p^2+q^2)y=qz.$

Solution :

Let $F(x,y,z,p,q)=(p^2+q^2)y-qz=0 \qquad ...(1)$

The subsidiary equations are:

$$\frac{dx}{-2py}=\frac{dy}{z-2qy}=\frac{dz}{-qz}=\frac{dp}{-pq}=\frac{dq}{p^2}$$

$pdp+qdq=0$

$p^2+q^2=c^2$

put $p^2+ç^2=c^2$ in equation (1) so that $q=\frac{c^2y}{z} \qquad ...(2)$

$p = c\sqrt{\frac{z^2-c^2y^2}{z}}$

Hence, $dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2 - c^2y^2)dx} + \frac{c^2y}{z}dy$

$=> zdz - c^2ydy = c\sqrt{z^2 - c^2y^2}dx$

$=> \frac{(1/2)d(z^2-c^2y^2)}{\sqrt{z^2-c^2y^2}} = cdx$

Integrating, we get the required solution as $z^2 = (a + cx)^2 + c^2y^2$

6.4 Applications to Wave Equation and Laplace's Equation

Charpit's method is not typically used to solve second-order partial differential equations (PDEs) like the wave equation and Laplace's equation directly. However, there are certain ways to relate Charpit's method to these equations indirectly or to solve problems related to them using characteristics, which are a central concept in Charpit's method. Let's explore these connections:

1. **Wave Equation**:

Characteristic Curves: Although Charpit's method is not directly applied to solve the wave equation, the concept of characteristic curves can still be useful. For example, in problems involving initial value conditions for the wave equation, characteristic curves represent the paths along which information (waves) propagate. Charpit's method could be used to analyze characteristics and their behavior in such problems, although it wouldn't lead to a direct solution of the wave equation itself.

2. **Laplace's Equation**:

Characteristic Surfaces: In Laplace's equation, characteristic surfaces are surfaces along which the solution is constant. Charpit's method could be applied to analyze characteristic surfaces and understand the behavior of the solution. However, this analysis wouldn't directly provide a solution to Laplace's equation but could offer insights into the nature of the solution and its behavior.

While Charpit's method may not offer a direct solution to the wave equation or Laplace's equation, its concepts, such as characteristics and characteristic curves/surfaces, can still be valuable for understanding the behavior of solutions to these equations in specific contexts. However, for direct solutions to these equations, other methods like separation of variables, Fourier methods, or Green's function techniques are typically employed.

6.5 Summary

A method for resolving particular types of ordinary differential equations (ODEs) is the Method of Undetermined Coefficients. Finding specific solutions to non-homogeneous linear differential equations is where it comes in most helpful.

6.6 Keywords

1. Non-homogeneous differential equations
2.Particular solution
3.Complementary solution
4. Linear differential equations
5. Undetermined coefficients6.Auxiliary equation

6.7 Self Assessment questions

1.What is the fundamental difference between homogeneous and non-homogeneous differential equations?
2.Explain why the Method of Undetermined Coefficients is particularly useful for solving nonhomogeneous linear differential equations.
3. When applying the Method of Undetermined Coefficients, why is it necessary to identify the form of the non-homogeneous term in the differential equation?
4 .Describe the process of assuming particular solution fom in the Method of Undetermined Coefficients. Why is this step important?
5. How do you determine the undetermined coefficients in the Method of Undetermined Coefficients ?What technique is commonly used for this purpose?

6.8 Case Study

Simulating Population Increase

Consider that you have been assigned the responsibility of researching the spread of a specific species in a remote area, such a small island. Your goal is to forecast the population's future evolution under specific scenarios.

Question: To describe the population expansion, you choose to employ a non-homogeneous linear differential equation. Three terms one for each of the death, birth, and carrying capacities of the environment will be featured in the equation.

6.9 References

1. William. E.(2017). Elementary Differential Equations and Boundary Value Problems. John Wiley & Sons.
2. Erwin Kreyszig (2020). Advanced Engineering Mathematics. John Wiley & Sons

UNIT 7

Separation of Variables

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure

7.1 Method of Separation of Variables

A powerful method i.e. the method of separation of variables of finding solutions of linear partial differential equations of order two with prescribed initial and boundary conditions is applicable in certain circumstances. In this method, we first assume a trial solution of the given partial differential equation in the form of a product of number of functions, each of which is a function of one independent variable alone. For this reason, this method is also known as the product method. The method of separation of variables can be best illustrated by means of a particular example as follows:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

be the general linear partial differential equation, where A,B,C,....,G are functions of x and y. Here, it must be noted that

(i) If any partial differential equation of order two cannot be put in the above form as given in equation , then it is said to be non-linear.

(ii) If we have G=0 in the above partial differential equation, then it is said to be homogeneous otherwise it is no homogeneous Thus, for non-homogeneous partial differential equation, we note that G not equal to 0.

1.Solve : Using method of separation of variables.

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

Solution:

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

$$u(x,y)=X(x)Y(y) \quad 1\ \frac{\partial u}{\partial x}=X'Y,\ \frac{\partial u}{\partial y}=XY' \text{ and } \frac{\partial^2 u}{\partial x^2}=X$$

$$X\text{ -2-}\lambda X= \text{ or } (D^2-2D-\lambda)X=0$$

$$Y^{'}-\lambda Y=0 \quad \text{or} \qquad \frac{dY}{dy}=\lambda Y$$

$$m^2-2m-\lambda=0, \text{ which gives } \quad m=1\pm\sqrt{(1+\lambda)}$$

Thus, we have $X(x)=Ae^{(1+\sqrt{1+\lambda})x}+Be^{(1-\sqrt{1+\lambda})x}$

$$Y(y)=Ce^{\lambda y}$$

$$u(x,y)=\left[ACe^{(1+\sqrt{(1+\lambda)})x}+BC\,e^{(1-\sqrt{(1+\lambda)})x}\right]e^{\lambda y}$$

$$u(x,y)=\left[A_1e^{(1+\sqrt{(1+\lambda)})x}+A_2e^{(1-\sqrt{(1+\lambda)})x}\right]e^{\lambda y}$$

7.2 Introduction and Overview

The method of separation of variables is a powerful technique commonly used to solve partial differential equations (PDEs), particularly those that are linear and homogeneous, with constant coefficients. This method exploits the assumption that the solution to the PDE can be expressed as a product of functions, each depending on only one of the independent variables. Let's delve into an introduction and overview of this method. Partial differential equations arise in various scientific and engineering disciplines to describe physical phenomena involving multiple independent variables. Solving these equations allows us to understand and predict the behavior of systems governed by them. The method of separation of variables provides a systematic approach to finding solutions to certain types of partial differential equations.

The method of separation of variables involves the following general steps:

1. **Formulation of the PDE**: Begin with a given partial differential equation (PDE) that needs to be solved. This equation typically describes the relationship between the dependent variable and its independent variables, along with any given boundary or initial conditions.

2. **Assumption of Separable Solution**: Assume that the solution to the PDE can be expressed as a product of functions, each depending on only one of the independent variables.
3. **Substitution and Separation**: Substitute the assumed separable solution into the original PDE and then separate the variables by moving all terms involving each variable to one side of the equation. This results in a set of ordinary differential equations (ODEs), each involving only one independent variable and its corresponding function.
4. **Solving the ODEs**: Solve each resulting ODE individually to find the solutions for the functions. This step usually involves finding eigen values and eigen functions or applying other techniques appropriate to the specific ODE.
5. **Combining Solutions**: After obtaining the solutions for each function, combine them using the assumed separable form to form the general solution to the original PDE.
6. **Applying Boundary or Initial Conditions**: If boundary or initial conditions are provided with the problem

7.3 Application to one-dimensional wave equation and diffusion equation

Consider a P.D.E

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \ 0 < t < \infty$$

with the boundary conditions (**BCs**):

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

and the initial condition (**IC**):

$$u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant L$$

7.4 Summary

By dissecting the issue into more manageable ordinary differential equations (ODEs), separation of variables is a potent strategy for solving partial differential equations (PDEs). The essential concept is to presume that the PDE's solution may be written as the product of functions, each of which depends only on one variable.

7.5 Keywords

1. Method of Separation
2. Product Solutions
3. Fourier Series
4. Laplace's Equation
5. Heat Equation
6. Wave Equation
7. Diffusion Equation

7.6 Self Assessment questions

1. Define Separation of Variables and explain its significance in solving partial differential equations.
2. Describe the general procedure for applying Separation of Variables to solve a partial differential equation.
3. What are the key assumptions made when applying Separation of Variables?
4. Provide an example of a partial differential equation problem where Separation of Variables can be effectively applied.
5. How do you determine the separated solutions for each variable in the Separation of Variables method?

7.7 Case Study

A non-mathematical context: designing a multi-stage distillation column for separating components in a chemical process.

Problem Statement:

Imagine a chemical engineering company tasked with designing a distillation column to separate a mixture of ethanol and water into its pure components. The goal is to achieve high purity ethanol as the top product and high purity water as the bottom product.

7.8 References

3. William. E.(2017). Elementary Differential Equations and Boundary Value Problems. John Wiley & Sons.
4. Erwin Kreyszig (2020). Advanced Engineering Mathematics. John Wiley & Sons

UNIT 8

PDEs of Second Order

Learning Objectives:

- Define partial differential equations (PDEs) of second order and distinguish them from other types of PDEs, such as first-order PDEs.
- Identify and derive the canonical forms of second-order PDEs, such as the Laplace, heat, and wave equations, in various coordinate systems.
- Discuss numerical techniques, such as finite difference, finite element, and spectral methods, for approximating solutions to PDEs.

Structure

8.1 PDEs of Second Order with Variable Coefficients

8.2 Introduction to Monge's Method

8.3 Solving PDEs with Variable Coefficients

8.4 Summary

8.5 Keywords

8.6 Self-Assessment Questions

8.7 Case Study

8.8 References

8.1 PDEs of Second Order with Variable Coefficients

Partial differential equations (PDEs) of second order with variable coefficients are equations that involve second-order partial derivatives of a function of several independent variables, where the coefficients of these derivatives may vary with respect to the independent variables. These equations are encountered in many areas of science and engineering, such as heat conduction, fluid dynamics, electromagnetism, and quantum mechanics. The general form of a second-order linear partial differential equation with variable coefficients is:

$$A(x,y,z)\frac{\partial^2 u}{\partial x^2} + B(x,y,z)\frac{\partial^2 u}{\partial x \partial y} + C(x,y,z)\frac{\partial^2 u}{\partial y^2} + D(x,y,z,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = 0$$

Where u is the unknown function of x and y, and A, B, C, and D are functions of x, y, and u, and their partial derivatives up to the first order.

Solving PDEs of this form can be challenging and often requires various techniques depending on the specific equation and problem at hand. Here are some common methods for solving second-order PDEs with variable coefficients.

$$f(x,y,z,p,q,r,s,t)=0$$

$$Rr+Ss+Tt+Pp+Qq+Zz=F$$

$$p\equiv\frac{\partial z}{\partial x},q\equiv\frac{\partial z}{\partial y},r\equiv\frac{\partial^2 z}{\partial x^2},s\equiv\frac{\partial^2 z}{\partial x\partial y},t\equiv\frac{\partial^2 z}{\partial y^2}$$

Classification of Linear Partial Differential Equations of Order Two with Variable Coefficients

Linear partial differential equations of order two with variable coefficients may be classified and categorized as follows:

Linear Partial Differential Equations with Variable coefficients Categorized as Type I.

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{F}{R}=F_1(x,y)$$

$$s=\frac{\partial^2 z}{\partial x \partial y}=\frac{F}{S}=F_2(x,y)$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{F}{T}=F_3(x,y)$$

1.Solve :

(i) $r=6x$ (ii) $r=\sin(xy)$

Solution: (i)

$$\frac{\partial^2 z}{\partial x^2}=6x$$

integrating w.r.t.x

$$\frac{\partial z}{\partial x}=3x^2+\phi_1(y)$$

where $\phi_1(y)$ is an arbitrary function of y.

$$z=x^3+x\phi_1(y)+\phi_2(y)$$

(ii) The given P.D.E. can be written as

$$\frac{\partial^2 z}{\partial x^2}=\sin(xy)$$

$$\frac{\partial z}{\partial x}=\frac{-1}{y}\cos(xy)+\phi_1(y)$$

$$z=\frac{-1}{y^2}\sin(xy)+x\,\phi_1(y)+\phi_2(y)$$

2.Solve:

(i) $t=\sin(xy)$ (ii) $t=x^2\cos(xy)$

Solution : (i)

$$\frac{\partial^2 z}{\partial y^2}=\sin(xy)$$

$$\frac{\partial z}{\partial y}=\frac{-1}{x}\cos(xy)+\phi_1(x)$$

$$z=\frac{-1}{x^2}\sin(xy)+y\,\phi_1(x)+\phi_2(x)$$

(ii) The given P.D.E can be written as

$$\frac{\partial^2 z}{\partial y^2}=x^2\cos(xy)$$

$$\frac{\partial z}{\partial y}=x\sin(xy)+\phi_1(x)$$

$$z=-\cos(xy)+y\,\phi_1(x)+\phi_2(x)$$

3.Solve:

$$xs+q=4x+2y+2.$$

Solution: The given P.D.E can be written as

$$x\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y}=4x+2y+2 \quad \text{or} \quad \frac{\partial}{\partial y}(xp+z)=4x+2y+2$$

$$xp+z=4xy+y^2+2y+\phi_1(x)$$

$$x\frac{\partial z}{\partial x}+z=4xy+y^2+2y+\phi_1(x)$$

$$\frac{\partial}{\partial x}(xz)=4xy+y^2+2y+\phi_1(x)$$

$$xz=2x^2y+xy^2+2xy+\int\phi_1(x)\,dx+\psi_2(y)$$

Exercise

1. $ar=xy$ **2.** $x=x^2e^y$ **3.** $r=2y^2$

4. $s=x-y$ **5.** $s=x^2-y^2$ **6.** $xys=1$

7. $x^2s=\sin y$ **8.** $s=\frac{x}{y}+a$ **9.** $t=\sin(xy)$

10. $2yq+y^2t=1$ **11.** $ys+p=\cos(x+y)-y\sin(x+y)$

Answers:

1. $z=\frac{x^3 y}{6a}+x\phi_1(y)+\phi_2(y)$

2. $z=\frac{x^4}{12}e^y+x\phi_1(y)+\phi_2 y$

3. $z=x^2 y^2+x\phi_1(y)+\phi_2(y)$

4. $z=\frac{1}{2}(x^2 y-x y^2)+\psi_1(y)+\psi_2(x)$

5. $z=\frac{1}{3}(x^3 y-x y^3)+\psi_1(y)+\psi_2(x)$

6. $z=\log x \log y+\psi_1(y)+\psi_2(x)$

7. $z=\frac{1}{x}\cos y+\psi_1(y)+\psi_2(x)$

8. $z=\frac{x^2}{2}\log y+axy+\psi_1(y)+\psi_2(x)$

9. $z=\frac{-1}{x^2}\sin(xy)+y\phi_1(x)+\phi_2(x)$

10. $z=\frac{-1}{y}\phi_1(x)+\phi_2(x)$

11. $yz=y\sin(x+y)+\psi_1(y)+\psi_2(y)$

8.2 Introduction to Monge's Method

A partial differential equation of order two with two independent variables, x and y, and a dependent variable, z, may be stated in its most generic form as

$$f(x,y,z,p,q,r,s,t)=0$$

$$p\equiv\frac{\partial z}{\partial x},\ q\equiv\frac{\partial z}{\partial y},\ r\equiv\frac{\partial^2 z}{\partial x^2},\ s\equiv\frac{\partial^2 z}{\partial x\partial y}\quad\text{and}\quad t\equiv\frac{\partial^2 z}{\partial y^2}$$

It is referred to as a non-linear partial differential equation of order two if any one or more of the partial derivatives r, s, or t exist in (1) in more than one degree.

Occasionally, equation (1) can be expressed as

$$Rr+Sr+Tt+Pp+Qq+Zz=F$$

where the coefficients R, S, T, P, Q, Z, and F are the variable coefficients and not all zero or constants.

Monge's Method for non-linear P.D.E (Rr+Ss+Tt=V)

Since z is a function of x and y and $p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, r=\frac{\partial^2 z}{\partial x^2}$

$s=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x}$ and $t=\frac{\partial^2 z}{\partial y^2}$; therefore, we can write

$$\left.\begin{aligned} r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial p}{\partial x}, s=\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial q}{\partial x}, \\ s=\frac{\partial^2 z}{\partial y \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial p}{\partial y}, t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial q}{\partial y} \end{aligned}\right\}$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$$

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy=rdx+sdy$$

$$dq=\frac{\partial q}{\partial x}dx+\frac{\partial q}{\partial y}dy=sdx+tdy$$

$$r=\frac{dp-s\,dy}{dx} \quad \text{and} \quad t=\frac{dq-sdx}{dy}$$

$$R\left(\frac{dp-s\,dy}{dx}\right)+Ss+T\left(\frac{dq-sdx}{dy}\right)=V$$

$$R(dp-s\,dy)dy+Ss\,dx\,dy+T(dq-s\,dx)d\,x=V\,dx\,dy$$

$$R\,dp\,dy+T\,dq\,dx-V\,dx\,dy]$$

$$-s\left[R(dy)^2-Sdx\,dy+T(dx)^2\right]=0$$

$$R\,dpdy+T\,dq\,dx-V\,dx\,dy=0$$

$$R(dy)^2-S\,dx\,dy+T(dx)^2=0$$

4.Solve

$r=a^2t.$

Solution:

$$r-a^2t=0$$

Comparing it with $Rr+Ss+Tt=V$, we have

$R=1,\ S=0,\ T=-a^2$ and $V=0.$

$R\,dp\,dy+T\,dq\,dx-Vdx\,dy=0$ i.e. $dp\,dy-a^2\,dq\,dx=0$

$R(dy)^2-S\,dx\,dy+T(dx)^2=0$ i.e. $(dy)^2-a^2(dx)^2=0$

$(dy-a\,dx)(dy+a\,dx)=0$

$$dy - a\,dx = 0$$

$$dy + a\,dx = 0$$

$$y - ax = c_1$$

$$dp - a\,dq = 0, \text{ since } \quad dy = a\,dx$$

$$p - aq = c_2$$

$$p - aq = \phi_1(y - ax)$$

$$p + aq = \phi_2(y + ax)$$

$$p = \frac{1}{2}\left[\phi_1(y - ax) + \phi_2(y + ax)\right]$$

$$q = \left(\frac{-1}{2a}\right)\left[\phi_1(y - ax) + \phi_2(y + ax)\right]$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{1}{2}\left[\phi_1(y - ax) + \phi_2(y + ax)\right] dx$$

$$-\left(\frac{1}{2a}\right)\left[\phi_1(y - ax) + \phi_2(y + ax)\right] \text{dy}$$

$$dz = \left(\frac{-1}{2a}\right)\phi_1(y - ax)[dy - adx] + \left(\frac{1}{2a}\right)\phi_2(y + ax)[dy + adx]$$

$$dz = \left(\frac{-1}{2a}\right)\phi_1(y - ax)[d(y - ax)] + \left(\frac{1}{2a}\right)\phi_2(y + ax)\,d(y + ax)$$

$$z = \psi_1(y - ax) + \psi_2(y + ax)$$

8.3 Solving PDEs with Variable Coefficients

Quasi-linear Partial Differential Equation of Order Two:-

A partial differential equation of order two of the form

Rr + Sr + Tt + Pp + Qq + Zz = F

Where R, S, T are functions of x,y,z, p and q is called a quasilinear partial differential equation of order two.

Semi-linear Partial Differential Equation of Order Two:-

A partial differential equation of order two of the form

Rr + Sr + Tt + Pp + Qq + Zz = F

Where R, S, T are functions of y and z only, is called a semi linear partial differential equation of order two.Linear and Non-linear Partial Differential Equations of Order Two with Variable Coefficients:-

A partial differential equation of order two of the form

$$Rr+Sr+Tt+Pp+Qq+Zz=F$$

5.Solve: t+s+q=0

Solution : The given P.D.E can be written as

$$\frac{\partial q}{\partial y}+\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y}=0$$

Integrating w.r.t. y, we get $q+p+z=f(x)$

$$p+q=f(x)-z$$

$$Pp+Qq=R.$$

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{f(x)-z}$$

I.F. $e^{\int dx} = e^x$

$$z e^x = \int e^x f(x)\, dx + c_2 \quad \text{or} \quad z e^x - \phi(x) = c_2$$

where $\phi(x) = \int e^x (x)\, dx$.

$$z e^x - \phi(x) = \psi(x - y) \quad \text{or} \quad z e^x = \phi(x) + \psi(x - y)$$

Exercise

1. $2xr - ys + 2p = xy^2$ **2.** $xr + ys + p = 10 x y^3$

3. $2yt - xs + 2q = 4x^2 y$ **4.** $yt + xs + q = 8x^2 y + 9y^2$

5. $z + r = x \cos(x + y)$ **6.** $sy - 2xr - 2p = 6xy$

Answer

1. $z=\phi_1(xy^2)+\phi_2(y)+\frac{x^2y^2}{4}$

2. $z=\phi_1\left(\frac{x}{y}\right)+\phi_2(y)+x^2y^3$

3. $z=\phi_1(x^2y)+\phi_2(x)+x^3y^2$

4. $z=\phi_1\left(\frac{x}{y}\right)+\phi_2(y)+x^2y^2+y^3$

5. $z=\phi_1(y)+\phi_2(y-x)-\frac{1}{2}\cos(x+y)+\frac{1}{4}\sin(x+y)$

6. $z=-x^2y+\phi_1(xy^2)+\phi_2(y)$

8.4 Summary

Understanding and solving second-order PDEs is crucial for modeling and analyzing complex physical systems, predicting their behavior, and designing optimal engineering solutions. These equations play a fundamental role in diverse areas of science and technology, contributing to advancements in fields ranging from aerospace engineering to medical imaging.

8.5 Keywords

1. Second-order PDEs
2. Elliptic equations
3. Parabolic equations
4. Hyperbolic equations
5. Boundary value problems (BVPs)

8.6 Self-Assessment Questions

1. What are the key characteristics of a partial differential equation of second order, and how do they differ from those of first-order PDEs?
2. Can you provide examples of physical phenomena or mathematical models described by second-order partial differential equations?
3. What are the main classifications of second-order PDEs, such as elliptic, parabolic, and

hyperbolic, and how do their properties differ?

4. How do boundary value problems (BVPs) and initial value problems (IVPs) arise in the context of second-order PDEs, and what techniques are used to solve them?
5. What role do boundary conditions and initial conditions play in determining solutions to second-order PDEs?
6. Can you explain the concept of characteristic curves or surfaces in the context of hyperbolic and parabolic second-order PDEs?
7. How do numerical methods, such as finite difference, finite element, and spectral methods, contribute to the solution of second-order PDEs?
8. What are some applications of second-order PDEs in various fields, such as physics, engineering, and finance?
9. How do nonlinear second-order PDEs differ from linear ones, and what challenges do they pose in terms of analysis and solution?
10. Can you describe the significance of eigenvalue problems associated with second-order PDEs and their applications in mathematical modeling?

8.7 Case Study

Heat conduction is a common phenomenon encountered in various engineering and scientific applications. The mathematical model describing heat conduction often involves partial differential equations (PDEs) of second order, specifically the heat equation. Understanding the behavior of temperature distribution in a homogeneous medium subjected to heat sources or boundary conditions is essential in various fields, including thermal engineering, materials science, and physics.

Objective:

To analyze the temperature distribution in a homogeneous medium undergoing heat conduction using the heat equation, a second-order partial differential equation, and to investigate the effects of different boundary conditions and heat sources.

8.8 References

1. Evans, L. C. (2010). Partial Differential Equations (2nd ed.). American Mathematical Society.

2. Strauss, W. A. (2008). Partial Differential Equations: An Introduction. John Wiley & Sons.

UNIT - 9

Geometrical Interpretation

Learning Objectives:

- Develop an understanding of fundamental geometric concepts such as points, lines, planes, curves, surfaces, and their properties.
- Enhance their ability to visualize geometric objects and transformations in two-dimensional and three-dimensional space.
- Investigate geometric properties such as distance, angle, area, volume, curvature, and symmetry, and understand their significance in various contexts.

Structure:

9.1 Total Differential Equations

Total differential equations, also known as total derivative equations, arise in mathematical modeling when considering how a function of several variables changes as all its variables change simultaneously. These equations involve total derivatives, which account for changes in the function with respect to all of its independent variables.

The general form of a total differential equation for a function

$$df = f_x(x, y, z) \cdot dx + f_y(x, y, z) \cdot dy + f_z(x, y, z) \cdot dz.$$

Total differential equations are often used in various fields, including physics, economics, engineering, and biology, to model systems where multiple variables interact and change together. For example:

- In thermodynamics, total differential equations are used to describe changes in the internal energy, enthalpy, or entropy of a system with changes in temperature, pressure, and volume.
- In economics, total differential equations can describe how quantities such as production, consumption, and prices change in response to changes in input factors like labor, capital, and technology.
- In fluid dynamics, total differential equations can describe how fluid properties such as velocity, pressure, and temperature change as the fluid flows through a system.
- In population dynamics, total differential equations can model how populations of different species change over time due to birth, death, immigration, and emigration.

Solving total differential equations typically involves integrating both sides of the equation with respect to the respective variables. Depending on the specific problem, additional constraints or boundary conditions may be applied to determine the solution uniquely. Overall, total differential equations provide a powerful tool for understanding and analyzing systems with multiple interacting variables and are widely used in mathematical modeling across various disciplines.

9.2 Definition and Forms

1. **First-Order Total Differential Equations**:

   These equations involve first-order total differentials. They are typically of the form:

   dz= A(x,y)dx+ B(x,y)dydz

   Where A and B are functions of x and y.

2. **Higher-Order Total Differential Equations**:

These equations involve higher-order total differentials. For example, a second-order total differential equation might take the form:

$$d^2z = A(x,y)dx^2 + 2B(x,y)dxdy + C(x,y)dy^2$$

Where A, B, and C are functions of x and y.

3. **Systems of Total Differential Equations**:

   These equations involve multiple functions and their total differentials. They can describe relationships between multiple variables and their rates of change. An example of a system of total differential equations is:

$$dz_1 = A(x,y)dx + B(x,y)dy$$
$$dz_2 = C(x,y)dx + D(x,y)dy$$

   Where z_1 and z_2 are functions of x and y, and A, B, C, and D are functions of x and y.

Total differential equations play a crucial role in various fields of science and engineering, including physics, economics, biology, and engineering. They are used to model systems where multiple variables interact, and understanding their behavior requires considering how all variables change together. Solving total differential equations often involves integrating both sides of the equation with respect to the respective variables and applying any additional constraints or boundary conditions given by the problem.

1. Solve: xy = c (a-z)

Solution:

$$\frac{xy}{a-z} = c$$

$$d\left(\frac{xy}{a-z}\right) = 0$$

$$\Rightarrow \quad \frac{(a-z)\, d(xy) - xy\, d(a-z)}{(a-z)^2} = 0$$

$$\Rightarrow \quad (a-z)\,[x\,dy + y\,dx] + xy\,dz = 0$$

2. Solve:

$$x^2 + y^2 + z^2 = xc$$

Solution:

$$\frac{x^2 + y^2 + z^2}{x} = c$$

$$d\left(\frac{x^2 + y^2 + z^2}{x}\right) = 0$$

$$\Rightarrow \quad d\left(\frac{x^2}{x}\right) + d\left(\frac{y^2}{x}\right) + d\left(\frac{z^2}{x}\right) = 0$$

$$\Rightarrow \quad dx + \frac{2xydy - y^2dx}{x^2} + \frac{2xzdz - z^2dx}{x^2} = 0$$

$$\Rightarrow \quad x^2dx + 2xydy - y^2dx + 2xzdz - x^2dx = 0$$

$$\Rightarrow \quad (x^2 - y^2 - z^2)dx + 2xydy + 2xzdz = 0$$

9.3 Solutions and Conditions

Let us consider the following D.E

$$3x^2(y + z)dx + (z^2 + x^3)dy + (2yz + x^3)dz = 0$$

$$(3xz + 2y)dx + xdy + x^2dz = 0$$

$$ydx + (z - y)dy + xdz = 0$$

$$f(x, y, z) = x^3y + x^3z + z^2y = c$$

$$d[x^3y + x^3z + z^3y] = 0$$

$$\Rightarrow 3x^2ydx + x^3dy + 3x^2zdx + x^3dz + z^2dy + 2zydz = 0$$

$$\Rightarrow 3x^2(y + z)dx + (x^3 + z^2)dy + (x^3 + 2zy)dz = 0$$

Theorem 1: A necessary and sufficient condition that the total differential equation Pdx + Qdy + Rdz = 0 is integrable is that

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

3. Solve:

$$(yz+2x)dx+(zx+2y)dy+(xy+2z)dz=0.$$

Solution:

$$P=(yz+2x),\ Q=(zx+2y),\ R=(xy+2z)$$

Now, $\frac{\partial P}{\partial y}=z=\frac{\partial Q}{\partial x}$

$$\frac{\partial Q}{\partial z}=x=\frac{\partial R}{\partial y}$$

$$\frac{\partial R}{\partial x}=y=\frac{\partial P}{\partial z}.$$

$$(yzdx+zxdy+xydz)+(2xdx+2ydy+2zdz)=0$$

$$d(xyz)+d(x^2+y^2+z^2)=0$$

$$xyz+x^2+y^2+z^2=c$$

9.4 Geometrical Interpretation and Examples

The geometrical interpretation of total differential equations provides insights into how changes in independent variables affect the dependent variable, both locally and globally. Let's explore this interpretation using a first-order total differential equation as an example:

dz=A(x,y)dx+B(x,y)dydz

In this equation, dz represents the change in the dependent variable z, while dx and dy represent small changes in the independent variables x and y, respectively. A(x,y) and B(x,y) are functions that describe how z changes with respect to changes in x and y, respectively.

Geometrical Interpretation:

1. **Slope Interpretation**:
2. **Directional Derivative Interpretation**:
3. **Vector Interpretation**:

By understanding the geometrical interpretation of total differential equations, we can gain insights into how changes in independent variables affect the dependent variable in various directions and understand the behavior of surfaces in multivariable calculus

9.5 Summary

Geometrical interpretation involves understanding fundamental geometric concepts such as points, lines, curves, and surfaces, and their properties in two-dimensional and three-dimensional space. Through visualization and analytical reasoning, students explore transformations, coordinate systems, and geometric properties like distance, angle, and curvature. Geometrical interpretation extends to diverse applications in science, engineering, and art, where geometric modeling, analysis, and visualization play crucial roles. By mastering geometrical interpretation, students develop problem-solving skills, critical thinking abilities, and an appreciation for the beauty and utility of geometry across various disciplines.

9.6 Keywords

- Geometrical Interpretation
- Curvature

9.7 Self-Assessment Questions

1. How does understanding geometric properties aid in solving real-world problems?
2. Can you explain the significance of coordinate systems in geometrical interpretation?
3. What are some common geometric transformations, and how do they affect geometric objects?
4. How does curvature influence the shape and behavior of curves and surfaces?

5. In what ways do geometric concepts intersect with other disciplines, such as physics or computer science?
6. How does visualization enhance our understanding of geometric relationships and structures?
7. What role does symmetry play in geometric interpretation and analysis?
8. How do we use geometric reasoning to prove theorems and solve geometric puzzles?
9. Can you provide examples of how geometric interpretation is applied in engineering or architecture?
10. What historical developments have shaped our understanding of geometry, and how do they influence modern applications?

9.8 Case Study

Geometric interpretation plays a fundamental role in computer graphics, where visual representations of objects and scenes are created and manipulated using mathematical models. From rendering lifelike images to simulating virtual environments, geometric interpretation enables the creation of immersive visual experiences in various applications, including gaming, animation, virtual reality, and computer-aided design (CAD).

Objective:

To explore how geometric interpretation is applied in computer graphics and its significance in creating realistic and interactive digital environments.

9.9 References

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UNIT - 10

Differential Equations of Second Degree

Learning Objectives:

- Define second degree differential equations and distinguish them from other types of differential equations.
- Classify second degree differential equations based on their coefficients and solutions.
- Analyze the behavior of solutions to second degree differential equations, including stability, periodicity, and asymptotic behavior.

Structure:

10.1 Total Differential Equations of Second Degree

A second-degree differential equation, often referred to as a second-order differential equation, involves the second derivative of an unknown function. These equations are fundamental in physics, engineering, and many other fields. Here, we'll explore the concept of total differential equations of the second degree and provide some insights into their forms, solutions, and applications.

General Form

A second-order differential equation generally has the form:

$$F(x, y, y', y'') = 0$$

Where:

- y is the unknown function of x,
- y' is the first derivative $\frac{dy}{dx}$,
- y'' is the second derivative $\frac{d^2y}{dx^2}$.

Linear Second-Order Differential Equations

Linear differential equations are a fundamental concept in mathematics and physics, particularly in the study of dynamical systems and modeling natural phenomena. A linear differential equation is an equation that is linear in the unknown function and its derivatives. It can be expressed in the general form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

Where y is the unknown function of x, y' denotes the derivative of y with respect to x, and g(x) is a given function of x. The coefficients $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ can be functions of x but are assumed to be continuous.

Types of Linear Differential Equations

Ordinary Linear Differential Equations (ODEs)

In the case where the unknown function depends on a single independent variable, x, the differential equation is called an ordinary linear differential equation (ODE). Ordinary differential equations arise in many areas of science and engineering, describing phenomena ranging from population growth to the behavior of electrical circuits.

Example:

$$y'' + p(x)y' + q(x)y = g(x)$$

Partial Linear Differential Equations (PDEs)

When the unknown function depends on multiple independent variables, the equation becomes a partial linear differential equation (PDE). These are often used to model physical systems where the behavior of the system depends on multiple spatial or temporal dimensions.

Example

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Methods for Solving Linear Differential Equations

1. **Method of Undetermined Coefficients:**

   This method is applicable to linear differential equations with constant coefficients and a right-hand side that is a polynomial, exponential, sine, cosine, or a linear combination of these functions.

2. **Variation of Parameters:**

   This method is used for solving non-homogeneous linear differential equations. It involves finding a particular solution by varying the parameters in the general solution of the corresponding homogeneous equation.

3. **Method of Integrating Factors:**

   This method is used to solve first-order linear ordinary differential equations. It involves multiplying both sides of the equation by an integrating factor to make the left-hand side a perfect differential, thus simplifying the integration process.

   Solved example:

1 . Solve (1+ y^2) dx=(tan^{-1}y –x) dy

Sol: Given equation is (1+ y^2) $\frac{dx}{dy}$ = (tan^{-1}y –x)

$$\frac{dx}{dy}+\left(\frac{1}{1+y^2}\right).\ x=\frac{\tan^{-1} y}{1+y^2}$$

It is the form of $\frac{dx}{dy}$ + p(y).x = Q(y)

$$\text{I.F} = e^{\int p(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{tan^{-1} y}$$

$$\Rightarrow \text{General solution is}\quad x.\ e^{tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2}.e^{\tan^{-1} y} dy + c$$

$$\Rightarrow x.\ e^{tan^{-1} y} = \int t.\ e^t\ dt + c$$

[put $tan^{-1} y = t$

$\Rightarrow \frac{1}{1+y^2} dy = dt$]

$$\Rightarrow x.\ e^{tan^{-1} y} = t\ .\ e^t - e^t + c$$

$$\Rightarrow x.\ e^{tan^{-1} y} = tan^{-1} y. e^{tan^{-1} y} - e^{tan^{-1} y} + c$$

$\Rightarrow x = tan^{-1} y - 1 + c/e^{tan^{-1} y}$ is the required solution

2. Solve (x+y+1) $\frac{dy}{dx}$ = 1.

Sol: Given equation is (x+y+1) $\frac{dy}{dx}$ = 1.

$$\Rightarrow \frac{dx}{dy} - x = y+1.$$

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

=> I.F $= e^{\int p(y)dy} = e^{-\int dy} = e^{-y}$

General solution is x X I.F = $\int Q(y) \times I.F.dy + c$

$=> x . e^{-y} = \int (1+y)\, e^{-y} dy + c$

$=> x . e^{-y} = \int e^{-y} dy + \int y e^{-y} dy + c$

$=> \quad x e^{-y} = - e^{-y} - y\, x e^{-y} . e^{-y} + c$

$=> \quad x e^{-y} = - e^{-y}(2+y) + c$ .//

3. Solve $y^1 + y = e^{e^x}$

Sol: Given equation is $y^1 + y = e^{e^x}$

It is of the form $\frac{dy}{dx} + p(x).y = \emptyset(x)$

Where p(x) =1 Q(x) = e^{e^x}

$=> \quad I.F = e^{\int p(x)dx} = e^{\int dx} = e^x$

General solution is y x I.F = $\int Q(x) \times I.F.dx + c$

$=> \quad y.\ e^x = \int e^{e^x} e^x dx + c$

$=> y.\ e^x = \int e^t dt + c$ $\quad \left\{ \begin{array}{l} \text{put } e^x = t \\ e^x dx = dt \end{array} \right\}$

$=> y.\ e^x = e^t + c$

$=> y.\ e^x = e^{e^x} + c$

10.2 Properties and Solutions

Second-order differential equations, particularly linear ones, have several properties and solution techniques that are fundamental in mathematical analysis and applications. Below, we will discuss the properties, general forms, and methods for solving second-order differential equations.

Properties of Second-Order Differential Equations

Second-order differential equations have several important properties that distinguish them from other types of differential equations. Here are some key properties:

1. **Second-Order Derivative:** Second-order differential equations involve the second derivative of the dependent variable (usually denoted as y or y(x). This means that the equation relates how the rate of change of the function's slope (acceleration) is related to the function itself.
2. **Linear or Nonlinear:** Second-order differential equations can be classified as linear or nonlinear. Linear second-order differential equations have a linear relationship between the function and its derivatives, while nonlinear equations involve nonlinear terms.
3. **Homogeneous or Non-Homogeneous:** Second-order differential equations can be homogeneous or non-homogeneous. Homogeneous equations have a non-zero right-hand side, while non-homogeneous equations have a zero right-hand side.
4. **Constant or Variable Coefficients:** The coefficients of the second-order derivative, first-order derivative, and the function itself can be constant or variable functions of the independent variable. Equations with constant coefficients are often easier to solve analytically.
5. **Boundary or Initial Conditions:** Solutions to second-order differential equations typically require boundary conditions (for boundary value problems) or initial conditions (for initial value problems) to uniquely determine the solution. These conditions specify the values of the function and/or its derivatives at specific points or over specific intervals.
6. **Existence and Uniqueness:** Under certain conditions, second-order differential equations have unique solutions satisfying given initial or boundary conditions. This

property is crucial for ensuring that solutions to physical problems are meaningful and predictable.

7. **Classification:** Second-order differential equations can be classified based on various criteria, such as their linearity, homogeneity, and coefficient functions. This classification helps in selecting appropriate solution techniques and understanding the behavior of solutions.
8. **Superposition Principle:** Linear second-order differential equations obey the superposition principle, meaning that the sum of any two solutions to the equation is also a solution. This property simplifies the solution process by allowing the construction of more complex solutions from simpler ones.

Example: Solve the following second-order differential equation:

$$y'' - 3y' + 2y = 0$$

Solution

1. Formulate the characteristic equation:

To solve the differential equation, we first convert it to its characteristic (auxiliary) equation. For the differential equation

$$y'' - 3y' + 2y = 0,$$

the characteristic equation is:

$$r^2 - 3r + 2 = 0$$

2. Solve the characteristic equation:

Factor the quadratic equation:

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0$$

Thus, the roots are:

$$r_1 = 1$$
$$r_2 = 2$$

3. Write the general solution:

Since the characteristic equation has two distinct real roots, the general solution to the differential equation is:

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Substituting the values of r_1 and r_2:

$$y(t) = C_1 e^{t} + C_2 e^{2t}$$

4. Apply initial conditions (if any):

If initial conditions are provided, say $y(0)=y_0$ and $y'(0)=y_0'$, we can use them to find C_1 and C_2.

Suppose $y(0)=3$ and $y'(0)=4y$.

Step 1: Apply $y(0) = 3$

$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2 = 3$

$C_1 + C_2 = 3$

Step 2: Apply $y'(0) = 4$

First, compute $y'(t)$:

$y'(t) = C_1 e^t + 2C_2 e^{2t}$

Then, at $t = 0$:

$y'(0) = C_1 e^0 + 2C_2 e^0 = C_1 + 2C_2 = 4$

$C_1 + 2C_2 = 4$

Step 3: Solve the system of equations:

$$\begin{cases} C_1 + C_2 = 3 \\ C_1 + 2C_2 = 4 \end{cases}$$

Subtract the first equation from the second:

$(C_1 + 2C_2) - (C_1 + C_2) = 4 - 3$

$C_2 = 1$

Substitute $C_2=1$ back into the first equation:

$$C_1 + 1 = 3$$
$$C_1 = 2$$

So, $C_1 = 2$ and $C_2 = 1$.

5. Write the particular solution:

Substitute C_1 and C_2 back into the general solution:

$$y(t) = 2e^t + e^{2t}$$

10.3 Applications and Examples

1. Mechanical Vibrations

Second-order differential equations are fundamental in describing the motion of mechanical systems subject to forces. Examples include:

- **Simple Harmonic Oscillator**: Models systems like springs and pendulums.
- **Damped Harmonic Oscillator**: Models systems with energy dissipation, such as a car shock absorber.
- **Forced Harmonic Oscillator**: Models systems subject to external periodic forces, like an AC circuit driven by a sinusoidal voltage.

2. Electrical Circuits

RLC circuits (Resistor, Inductor, Capacitor circuits) are modeled using second-order differential equations:

- **Series RLC Circuit**: The voltage across the circuit can be described by a second-order differential equation involving the resistance, inductance, and capacitance.

3. Population Dynamics

Models in biology often involve second-order differential equations to describe population dynamics with respect to time, considering factors like birth rates, death rates, and carrying capacities.

4. Heat Transfer and Diffusion

Heat conduction in solids and the diffusion of substances are often modeled using second-order partial differential equations like the heat equation and the diffusion equation.

5. Quantum Mechanics

The Schrödinger equation, a fundamental eq^n in quantum mechanics, is a second-order PDEs that depicts how the quantum state of a physical system modify over time.

10.4 Summary

Applications of second-degree differential equations are extensive. They are used to model various physical phenomena such as oscillations, vibrations, heat conduction, fluid flow, and electrical circuits. In engineering, they are fundamental in designing control systems, analyzing structural dynamics, and predicting system behavior. In biology, second-degree differential equations are used to model population dynamics, biochemical reactions, and physiological processes.

Understanding second-degree differential equations and their solutions is essential for gaining insights into the behavior of dynamical systems and for solving real-world problems across diverse scientific and engineering disciplines.

10.5 Keywords

1. Second-order differential equations
2. Homogeneous equations
3. Nonhomogeneous equations
4. Linear differential equations
5. Nonlinear differential equations

10.6 Self-Assessment Questions

1. What is the general form of a second-degree ordinary differential equation?
2. How do you classify second-degree ordinary differential equations based on their

coefficients?

3. Explain the characteristic equation associated with second-degree linear homogeneous differential equations.
4. What is the difference between a second-degree linear and a second-degree nonlinear ordinary differential equation?
5. How do you solve a second-degree linear homogeneous ordinary differential equation with constant coefficients?
6. Provide an example of a physical phenomenon modeled by a second-degree ordinary differential equation.
7. What role do boundary conditions play in solving second-degree partial differential equations?
8. How do you recognize a second-degree ordinary differential equation with variable coefficients?
9. Discuss the importance of initial conditions when solving second-degree ordinary differential equations.
10. Explain how to identify singular points in a second-degree ordinary differential equation.

10.7 Case Study

Differential equations of second degree are fundamental in describing the dynamics of mechanical systems, particularly in vibration analysis. In this case study, we'll explore the vibrations of a mass-spring system governed by a second-order ordinary differential equation (ODE).

Objective: To analyze the vibrations of a mass-spring system and determine the system's natural frequency and mode shapes using differential equations of second degree.

10.8 References

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UNIT - 11

Nonlinear Differential Equations

Learning Objectives:

- Define what nonlinear ordinary differential equations (ODEs) are and how they differ from linear ODEs.
- Classify nonlinear ODEs based on their order, degree, and form.
- Explore conditions under which solutions may exist and be unique, including Lipschitz continuity and the Picard-Lindelöf theorem.

Structure:

11.1 Series Solutions for Nonlinear Differential Equations

Series solutions for nonlinear differential equations are typically more complex than those for linear differential equations. While linear equations often allow for straightforward substitutions leading to linear recurrence relations, nonlinear equations require more sophisticated techniques. Here's a general approach to finding series solutions for nonlinear differential equations:

Step 1: Identify the Equation and the Series Solution

Identify the nonlinear differential equation you want to solve and assume a series solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Step 2: Differentiate the Series

Differentiate y(x) with respect to x to obtain expressions for y′(x), y″(x), and higher derivatives.

Step 3: Substitute the Series and Its Derivatives into the Equation

Substitute y(x), y′(x), y″(x), and higher derivatives into the nonlinear differential equation.

Step 4: Expand and Collect Terms

Expand the series and collect terms with the same powers of x.

Step 5: Solve the Recurrence Relation

Equating coefficients of like powers of xxx to zero, you'll get a recurrence relation involving the coefficients a_n.

1.Solve:

Find a series solution the nonlinear differential equation:

$$y''(x) - xy(x) + (y(x))^2 = 0$$

Solution:

Step 1: Identify the Equation and the Series Solution

Given the nonlinear differential equation, we assume a series solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Step 2: Differentiate the Series

Differentiate y(x) with respect to xxx to obtain expressions for y′(x) and y″(x).

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Step 3: Substitute the Series and Its Derivatives into the Equation

Substitute y(x), y′(x), and y″(x) into the differential equation:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n + \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = 0$$

Step 4: Expand and Collect Terms

Expand the series and collect terms:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} + \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = 0$$

Step 5: Solve the Recurrence Relation

Equating coefficients of like powers of x to zero:

$$(n+2)(n+1)a_{n+2} - a_n + \sum_{k=0}^{n} a_k a_{n-k} = 0$$

This equation represents a nonlinear recurrence relation involving the coefficients a_n.

11.2 Introduction to Nonlinear ODEs

Nonlinear ordinary differential equations (ODEs) are differential equations that contain nonlinear terms. In contrast to linear ODEs, where the dependent variable and its derivatives appear linearly, nonlinear ODEs can involve products, powers, or compositions of the dependent variable and its derivatives. Nonlinear ODEs arise in various fields of science and engineering, including physics, biology, chemistry, economics, and engineering.

Form of Nonlinear ODEs

A general form of a first-order nonlinear ODE is:

$$\frac{dy}{dx} = f(x, y)$$

Where y is the dependent variable, x is the independent variable, and f(x, y) is a nonlinear function of both x and y.

Second-order nonlinear ODEs can be expressed as:

$$F(x, y, y', y'') = 0$$

Where F is a nonlinear function of x,y,y′ and y″.

Types of Nonlinear ODEs

Nonlinear ODEs can take many forms, including:

1. **Algebraic Nonlinear ODEs**: The dependent variable and its derivatives are involved in algebraic nonlinear terms.
2. **Transcendental Nonlinear ODEs**: Nonlinear functions of the dependent variable and its derivatives contain transcendental functions such as exponential, logarithmic, trigonometric, or hyperbolic functions.
3. **Implicit Nonlinear ODEs**: The dependent variable and its derivatives appear implicitly in the equation, making it challenging to express the solution explicitly.

11.3 Methods for Finding Series Solutions

Finding series solutions for ordinary differential equations (ODEs) involves expressing the solution as a power series and determining the coefficients of the series. There are several methods for finding series solutions, each suited to different types of equations and boundary conditions. Here are some common methods:

1. Method of Undetermined Coefficients:

- This method is applicable when the differential equation is linear with constant coefficients and the non-homogeneous term is a polynomial, exponential, sine, cosine, or a combination of these functions.
- Assume a power series solution and substitute it into the differential equation. Equate coefficients of like powers of x to zero and solve for the undetermined coefficients.

2. Power Series Method:

- This method involves directly substituting a power series solution into the differential equation and solving for the coefficients.
- Useful for finding series solutions for linear and nonlinear differential equations when other methods are not applicable.

1. Method of Undetermined Coefficients:

The method of undetermined coefficients is a powerful technique used to find particular solutions to non-homogeneous linear differential equations, especially when the non-homogeneous term has a simple form such as polynomials, exponentials, sine, cosine, or combinations of these functions. It's particularly effective for constant coefficient linear differential equations.

Here's a step-by-step guide to applying the method of undetermined coefficients:

Step 1: Form of the Particular Solution

Based on the form of the non-homogeneous term F(x), guess the form of the particular solution $y_p(x)$. Common forms include:

- If F(x) is a polynomial of degree nnn, guess a polynomial of degree n with undetermined coefficients.
- If F(x) is e^{ax}, guess Ae^{ax}.
- If F(x) is $Asin(\omega x)+Bcos(\omega x)$, guess $Asin(\omega x)+Bcos(\omega x)$.
- If F(x) is a combination of these functions, guess a combination of the corresponding particular solutions.

Step 2: Substitute into the Differential Equation

Substitute the guessed particular solution y_p(x into the non-homogeneous differential equation and its derivatives.

Step 3: Solve for Undetermined Coefficients

Equating coefficients of like terms on both sides of the equation, solve for the undetermined coefficients.

Step 4: Include Homogeneous Solution

Combine the particular solution $y_p(x)$ with the complementary (homogeneous) solution $y_c(x)$ to get the general solution y(x).

Step 5: Apply Initial or Boundary Conditions

If initial conditions or boundary conditions are given, use them to determine the values of the undetermined coefficients in the particular solution.

2.Solve :

Solve the second-order linear non-homogeneous differential equation by Method of Undetermined Coefficients

$$y''(x) + y'(x) - 2y(x) = 3e^x$$

Solution:

Step 1: Guess the Form of the Particular Solution

Since the non-homogeneous term is $3e^x$, guess $y_p(x) = A e^x$

Step 2: Substitute into the Differential Equation

Substitute $y_p(x)$ and its derivatives into the differential equation:

$$A e^x - A e^x - 2 A e^x = 3 e^x$$

Step 3: Solve for Undetermined Coefficients

Equating coefficients of like terms:

$$-2A = 3$$

$$A = -3/2$$

Step 4: Include Homogeneous Solution

The complementary solution for the given homogeneous equation y′′(x)+y′(x)−2y(x)=0 can be found separately.

Step 5: Apply Initial or Boundary Conditions

If initial conditions or boundary conditions are given, use them to determine the value of the constant A.

2. Power Series Method:

The power series method is a technique used to find solutions to ordinary differential equations (ODEs) by expressing the solution as an infinite series of powers of the independent variable. This method is particularly useful for finding solutions to both linear and nonlinear ODEs, especially when other techniques are not applicable.

Here's a step-by-step guide to applying the power series method:

Step 1: Assume a Power Series Solution

Assume that the solution to the given differential equation can be expressed as a power series:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

a where a_n are coefficients to be determined and xxx is the independent variable.

Step 2: Differentiate the Series

Differentiate the assumed power series solution term by term to find expressions for y′(x), y′′(x), and higher-order derivatives $y^{(n)}$ (x).

Step 3: Substitute into the Differential Equation

Substitute the assumed power series solution and its derivatives into the given differential equation.

Step 4: Expand and Collect Terms

Expand the series and collect terms with the same powers of x on both sides of the equation.

Step 5: Equate Coefficients to Zero

Equate coefficients of like powers of x to zero to obtain a recurrence relation involving the coefficients a_n.

Step 6: Solve the Recurrence Relation

Solve the recurrence relation to find expressions for the coefficients a_n.

Step 7: Determine the Radius of Convergence

Analyze the convergence of the power series solution to determine the radius of convergence. The power series solution is valid within the radius of convergence.

3.Solve

Consider the second-order linear differential equation:

$$y''(x) + y(x) = 0$$

Solution:

Step 1: Differentiate the Series

Differentiate y(x) term by term to find expressions for y′(x) and y″(x):

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Step 2: Substitute into the Differential Equation

Substitute y(x), y′(x), and y″(x) into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 3: Expand and Collect Terms

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 4: Equate Coefficients to Zero

Equate coefficients of like powers of x to zero:

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

Step 5: Solve the Recurrence Relation

Solve the recurrence relation for the coefficients a_n:

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

Step 6: Determine the Radius of Convergence

Analyze the convergence of the power series to determine the radius of convergence.

Step 7: Sum the Series

Sum the power series to obtain the solution y(x).

11.4 Cauchy-Euler Equations and Special Cases

Cauchy-Euler equations, also known as Euler-Cauchy equations, are a type of linear homogeneous ordinary differential equation with variable coefficients. They are characterized by their form, which involves terms of the form x^r where r is a real or complex number. The general form of a Cauchy-Euler equation is:

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \ldots + a_1 x y'(x) + a_0 y(x) = 0$$

Where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants and y(x) is the unknown function to be solved for.

Special Cases of Cauchy-Euler Equations:

1. Constant Coefficients (Euler's Equation):

- When all coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants (i.e., a_i are not functions of x), the Cauchy-Euler equation reduces to Euler's equation.
- Euler's equation has the form

$$x^n y^{(n)}(x) + p_{n-1} x^{n-1} y^{(n-1)}(x) + \ldots + p_1 x y'(x) + p_0 y(x) = 0,$$

Where p_i are constants.

2. Linear Homogeneous Equations:

- When all coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, the Cauchy-Euler equation is a special case of a linear homogeneous ODE.
- The solution can be found using methods such as the characteristic equation or power

series methods.

3. Equations with Real Roots:

- When the characteristic equation $r(r-1)+p_1r+p_0=0$ has real roots $r_1,r_2,\ldots,r_n$, the general solution is given by

$$y(x) = c_1x^{r_1} + c_2x^{r_2} + \ldots + c_nx^{r_n}.$$

4. Equations with Repeated Roots:

- When the characteristic equation has repeated roots, the general solution includes terms of the form x^r and $x^r \ln(x)$.
- The general form of the solution can be found using the method of reduction of order.

5. Equations with Complex Roots:

- When the characteristic equation has complex roots $\alpha\pm\beta i$, the general solution includes terms of the form

$$x^\alpha \cos(\beta \ln(x)) \text{ and } x^\alpha \sin(\beta \ln(x)).$$

- Complex roots always occur in conjugate pairs for real coefficients.

4.Solve:

Consider the Cauchy-Euler equation:

$$x^2y''(x) - 3xy'(x) + 4y(x) = 0$$

Solution: This equation has the form

$$a_2x^2y''(x) + a_1xy'(x) + a_0y(x) = 0$$

with $a_2=1$, $a_1=-3$, and $a_0=4$.

The characteristic equation is

$$r(r-1) - 3r + 4 = 0,$$

which can be solved to find the roots $r_1=1$ and $r_2=4$. Therefore, the general solution is:

$$y(x) = c_1x^1 + c_2x^4$$

11.5 Summary

- Nonlinear differential equations represent a rich area of study with diverse applications and complex behavior.

- Understanding and analyzing these equations are essential for modeling and predicting the behavior of dynamical systems in various scientific and engineering disciplines.
- While solving nonlinear systems analytically is often challenging, a combination of qualitative, numerical, and analytical techniques enables researchers to gain insights into the behavior of nonlinear systems and develop effective strategies for prediction and control.

11.6 Keywords

- Nonlinearity
- Ordinary Differential Equations
- Partial Differential Equations
- Dynamical Systems
- Chaos Theory

11.7 Self-Assessment Questions

1. What distinguishes a nonlinear differential equation from a linear one?
2. Why are nonlinear differential equations often more challenging to solve than linear ones?
3. Give an example of a physical phenomenon that can be modeled by a nonlinear differential equation.
4. How does the presence of nonlinear terms affect the behavior of solutions to differential equations?
5. What are some common techniques for analyzing nonlinear differential equations?
6. Explain the concept of stability in the context of solutions to nonlinear differential equations.
7. How do numerical methods play a role in solving nonlinear differential equations?
8. Describe the significance of bifurcations in the study of nonlinear differential equations.
9. Give an example of a well-known nonlinear differential equation and its applications.
10. Discuss the importance of qualitative analysis in understanding solutions to nonlinear differential equations.

11.8 Case Study

Nonlinear differential equations play a vital role in modeling complex systems where interactions between variables are nonlinear. In this case study, we'll explore a population dynamics model to illustrate the application of nonlinear differential equations in ecology.

Question: Develop a population dynamics model using nonlinear differential equations and analyze the behavior of the population over time.

11.9 References

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2. Hale, J. K., &Koçak, H. (1991). Dynamics and Bifurcations. Springer.

UNIT - 12

Singular Point

Learning Objectives:

- Distinguish between regular points and singular points.
- Understand the different types of singular points, such as ordinary, regular, irregular, and essential singular points.
- Explore the behavior of solutions near different types of singular points.

Structure:

12.1 Solution near a Regular Singular Point

Consider a second-order ordinary differential equation of the form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

Suppose this equation has a regular singular point at $x=x_0$. We can approach this problem using the Frobenius method, which assumes a solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}$$

Here, r is the degree of singularity at the point x_0.

1. Substitute the series into the differential equation:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(x-x_0)^{n+r-2} + P(x)\sum_{n=0}^{\infty} a_n(n+r)(x-x_0)^{n+r-1} + Q(x)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+r} = 0$$

2. Simplify and collect terms:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(x-x_0)^{n+r-2} + \sum_{n=0}^{\infty} a_n P(x)(n+r)(x-x_0)^{n+r-1} + \sum_{n=0}^{\infty} a_n Q(x)(x-x_0)^{n+r} = 0$$

3. Equating coefficients of like powers of $(x-x_0)$ to zero:

$$\sum_{n=0}^{\infty} a_n[(n+r)(n+r-1) + P(x)(n+r) + Q(x)](x-x_0)^{n+r} = 0$$

This gives us a recurrence relation for the coefficients a_n:

$$(n+r)(n+r-1) + P(x)(n+r) + Q(x) = 0$$

4. Find the radius of convergence:

This recurrence relation can be used to determine the coefficients a_n. Once the coefficients are determined, we can find the radius of convergence of the series solution.

5. Analyze the behavior of the solution:

Depending on the behavior of P(x) and Q(x) near the singular point x_0, the solution may converge within a finite interval or may have an infinite radius of convergence.

1.Solve : Consider the Airy equation:

$$y''(x) - xy(x) = 0$$

The regular singular point of this equation is at x=0.

Solution: We assume the solution has the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Substitute into the Differential Equation

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

Equating coefficients of x^{n+r} to zero, we get:

$$(n+r)(n+r-1)a_n - a_{n+2} = 0$$

Since x=0 is a regular singular point, we'll look for solutions of the form r=α+β, where α and β are real numbers. Substituting into the equation above:

$$(\alpha+\beta+n)(\alpha+\beta+n-1)a_n - a_{n+2} = 0$$

$$(\alpha+\beta+n)(\alpha+\beta+n-1) - 0 = 0$$

$$(\alpha+\beta+n)(\alpha+\beta+n-1) = 0$$

This gives us two roots for α+β:

1. $\alpha + \beta = 0$
2. $\alpha + \beta = -1$

Case 1: $\alpha + \beta = 0$

Let's set $\alpha = 1/2$ and $\beta = -1/2$:

$$(1/2 + n)(-1/2 + n)a_n - a_{n+2} = 0$$

$$(n^2 - 1/4)a_n - a_{n+2} = 0$$

Solving for a_{n+2}:

$$a_{n+2} = \frac{n^2-1/4}{(n+2)(n+1)}a_n$$

Case 2: $\alpha + \beta = -1$

Let's set $\alpha = -1/2$ and $\beta = 1/2$:

$$(-1/2 + n)(1/2 + n)a_n - a_{n+2} = 0$$

$$(n^2 - 1/4)a_n - a_{n+2} = 0$$

12.2 Method of Frobenius

a The method of Frobenius is a technique used to find power series solutions for second-order linear differential equations with regular singular points. It's particularly useful when standard methods like variation of parameters or undetermined coefficients fail to yield solutions. Here's a step-by-step guide to applying the Frobenius method:

Step 1: Identify the Regular Singular Point

First, identify the point $x=x_0$ where the coefficient functions in the differential equation become singular but the equation remains well-defined. For a second-order linear differential equation of the form:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

if P(x) and Q(x) have poles at x_0 is not an ordinary point (where P(x) and Q(x) are both analytic), then x_0 is a regular singular point.

Step 2: Assume a Power Series Solution

Assume a solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$$

where r is a constant to be determined. This is called a Frobenius series.

Step 3: Substitute into the Differential Equation

Substitute the assumed solution into the differential equation and differentiate term by term to obtain expressions involving y(x), y′(x), and y″(x).

Step 4: Collect Terms and Equate Coefficients

Collect terms with the same powers of $x - x_0$ and equate coefficients to zero. This will yield a recurrence relation involving the coefficients a_n and possibly r.

Step 5: Solve for r

Solve for the values of r that satisfy the recurrence relation. Typically, this results in a quadratic equation in r.

Step 6: Analyze Solutions Based on r

Depending on the roots of the quadratic equation for r, there are three cases:

- If the roots are distinct, the series solution involves two linearly independent solutions.
- If the roots are repeated, additional logarithmic terms may appear in the series solution.
- If the roots are complex, the series solution involves complex exponentials.

Step 7: Solve for Coefficients and Determine Radius of Convergence

Solve the recurrence relation for the coefficients a_n and determine the radius of convergence of the power series solution.

Step 8: Construct the General Solution

Construct the general solution of the differential equation using the power series solutions obtained.

Step 9: Apply Initial or Boundary Conditions

Apply any given initial or boundary conditions to determine the values of arbitrary constants in the general solution.

The Frobenius method is particularly useful in solving differential equations with irregular behavior at singular points, leading to solutions expressed in terms of special functions such as Bessel functions, Legendre functions, or hyper geometric functions.

2.Solve: Bessel's Equation

Consider the Bessel's equation:

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$$

This equation has a regular singular point at x=0.

Solution:

Step 1: Assume a Power Series Solution

We assume a solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Step 2: Substitute into the Differential Equation

Substituting y(x), y′(x), and y″(x) into the differential equation, we get:

$$x^2 \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} + x\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + (x^2 - \nu^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Step 3: Collect Terms and Equate Coefficients

Equating coefficients of x^{n+r} to zero:

$$(n+r)(n+r-1)a_n + (n+r)a_n + (x^2 - \nu^2)a_n = 0$$

$$(n+r)^2 a_n - \nu^2 a_n = 0$$

Step 4: Solve for r

Setting $r(r-1) + r - \nu^2 = 0$:

$$r^2 - r + r - \nu^2 = 0$$

$$r^2 - \nu^2 = 0$$

$$r = \pm\nu$$

Step 5: Analyze Solutions Based on r

Since the roots are distinct, the Frobenius series has two linearly independent solutions. We will focus on one of them.

Step 6: Construct the Series Solution

For $r = \nu$, we have:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\nu}$$

Step 7: Solve for Coefficients and Determine Radius of Convergence

Substituting r=ν into the recurrence relation, we get:

$$\nu^2 a_n - \nu^2 a_n = 0$$

$$a_n = 0$$

This implies that all coefficients are zero except a_0, so the solution is:

$$y(x) = a_0 x^{\nu}$$

The radius of convergence of this series is infinite.

Step 8: Construct the General Solution

$$y(x) = C_1 J_{\nu}(x) + C_2 J_{-\nu}(x)$$

Where C_1 and C_2 are constants and $J_\nu(x)$ is the Bessel function of the first kind of order ν.

12.3 Finding Particular Integrals

Finding particular integrals involves finding a specific solution to a non-homogeneous linear differential equation. There are several methods to find particular integrals depending on the form of the non-homogeneous term. One common approach is the method of undetermined coefficients, especially when the non-homogeneous term is a polynomial, exponential, sine, cosine, or a combination of these functions.

Here's a step-by-step guide to finding particular integrals using the method of undetermined coefficients:

Step 1: Identify the Non-Homogeneous Term

Given a linear differential equation of the form:

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_1(x)y'(x) + a_0(x)y(x) = F(x)$$

Identify F(x) as the non-homogeneous term on the right-hand side.

Step 2: Guess the Form of the Particular Integral

Based on the form of F(x), guess a particular solution that matches the form of F(x). For example:

- If F(x) is a polynomial of degree m, guess a polynomial of degree m.
- If F(x) is e^{ax}, guess Ae^{ax}.
- If F(x) is sin(ax) or cos(ax), guess Asin(ax)+Bcos(ax).
- If F(x) is a combination of these functions, guess a combination of the corresponding particular solutions.

Step 3: Substitute the Guess into the Differential Equation

Substitute the guessed particular solution into the original differential equation and its derivatives.

Step 4: Solve for Undetermined Coefficients

Solve for the undetermined coefficients in the guessed particular solution by equating coefficients of like terms on both sides of the equation.

Step 5: Verify the Solution

Substitute the found particular solution back into the original differential equation to verify that it satisfies the equation. If it does, you've found the particular integral.

3.Solve:

Consider the differential equation:

$$y'' - y = \sin(x)$$

Solution:

Step 1: Identify the Non-Homogeneous Term

The non-homogeneous term is sin(x).

Step 2: Guess the Form of the Particular Integral

Since sin(x) is a trigonometric function, guess a particular solution of the form

$$y_p(x) = Asin(x) + Bcos(x)$$

Step 3: Substitute the Guess into the Differential Equation

$$y_p'' - y_p = -Asin(x) - Bcos(x) - (Asin(x) + Bcos(x))$$

Step 4: Solve for Undetermined Coefficients

Equating coefficients of sin(x) and cos(x), we get:

- $-A-A=1$ for sin(x), which gives $A=-1/2$.
- $-B-B=0$ for cos(x), which gives B=0.

Step 5: Verify the Solution

Substitute $A = -1/2$ and B=0 back into $y_p(x)$ and verify that it satisfies the differential equation. So, the particular integral is $y_p(x) = -12 \sin(x)$.

12.4 Summary

In summary, singular points are essential features of differential equations that provide valuable information about the behavior of solutions. Understanding their properties and employing appropriate techniques for their analysis are fundamental aspects of solving differential equations and studying physical phenomena.

12.5 Keywords

- Singular Point
- Irregular Singular Point
- Regular Singular Point
- Point of Singularity
- Removable Singularity

12.6 Self-Assessment Questions

1. What is a singular point in the context of differential equations?
2. How do singular points affect the solutions of differential equations?
3. Can a singular point be a regular point as well? Explain.
4. What are the types of singular points commonly encountered in differential equations?
5. How are singular points classified based on their behavior in the solution space?
6. Provide an example of a differential equation with a regular singular point.
7. Explain the concept of an irregular singular point.
8. How do singular points influence the convergence of power series solutions?
9. What is the significance of the Frobenius method in solving differential equations with singular points?
10. How do singular points relate to the behavior of solutions near critical points in dynamical systems?

12.7 Case Study

Singular points are critical in the study of ordinary differential equations (ODEs) and partial differential equations (PDEs). These points often indicate special behavior, such as the presence of a discontinuity or a breakdown in the regularity of the solution. Understanding singular points is crucial for analyzing the behavior of physical systems and developing appropriate mathematical models.

Objective: To investigate the nature and significance of singular points in differential equations through a case study involving a physical system.

12.8 References

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